Continuity of Drazin and generalized Drazin inversion in Banach algebras

by

Ronalda Abigail Marsha Benjamin

Thesis presented in partial fulfilment of the requirements for the degree of Master of Science in the Faculty of Science at Stellenbosch University

Supervisor: Prof. Sonja Mouton

March 2013

Declaration

By submitting this thesis electronically, I declare that the entirety of the work contained therein is my own, original work, that I am the sole author thereof (save to the extent explicitly otherwise stated), that reproduction and publication thereof by Stellenbosch University will not infringe any third party rights and that I have not previously in its entirety or in part submitted it for obtaining any qualification.

Date: ....

Abstract

A generalized inverse of an element in some algebraic structure satisfies appropriate modifications of the inverse of that element. Over the past decades, many authors have proposed and investigated various types of generalized inverses in a number of algebraic structures. In this thesis we devote our attention to the study of a generalized inverse introduced by M. P. Drazin, and a generalization thereof due to J. J. Koliha.

Denote by $\mathrm{N}(A)$ the set of all nilpotent elements of an algebra A, and by $\mathrm{QN}(A)$ the set of all quasinilpotent elements of a Banach algebra A. In 1958 Drazin introduced the concept of a Drazin inverse, then called a pseudo-inverse, in associative rings and semigroups. For the purpose of this thesis we will study this concept in the context of a general algebra. If A denotes an algebra, then we call an element $b \in A$ a Drazin inverse of $a \in A$ if

$$ab = ba, \;\; b = bab \text{ and } a - aba \in \mathrm{N}(A).$$

Years later Koliha generalized Drazin's definition as follows: Assuming A is a Banach algebra, we call an element $b \in A$ a generalized Drazin inverse of $a \in A$ if

$$ab = ba, \;\; b = bab \text{ and } a - aba \in \mathrm{QN}(A).$$

Since the Drazin (generalized Drazin) inverse is unique, denote by $a^d(a^D)$ the Drazin (generalized Drazin) inverse of a Banach algebra element a, and let $A^d(A^D)$ denote the set of Drazin (generalized Drazin) invertible elements in a Banach algebra A.

It is well-known that inversion is continuous on the set of invertible elements. In this thesis we provide conditions under which the maps $a \mapsto a^d$ and $a \mapsto a^D$ are continuous on A^d and A^D, respectively. Finally, we apply these results to the Banach algebra of bounded linear operators.

Opsomming

’n Veralgemeende inverse van ’n element in ’n sekere algebraïese struktuur voldoen aan geskikte wysigings van die inverse van daardie element. Oor die afgelope dekades het baie outeurs verskeie tipes veralgemeende inverses in ’n aantal algebraïese strukture voorgestel en ondersoek. In hierdie tesis fokus ons op ’n veralgemeende inverse bekendgestel deur M. P. Drazin, en ’n veralgemening daarvan toegeskryf aan J. J. Koliha.

Laat $\mathrm{N}(A)$ die versameling van nulpotente elemente in ’n algebra A wees en $\mathrm{QN}(A)$ die versameling van kwasi-nulpotente elemente in ’n Banach algebra A. In 1958 het Drazin die konsep van ’n Drazin inverse, eers genoem ’n pseudo-inverse, voorgestel in assosiatiewe ringe en semigroepe. Vir die doel van hierdie tesis sal hierdie konsep in die konteks van ’n algemene algebra bestudeer word. As A ’n algebra is, dan word ’n element $b \in A$ ’n Drazin inverse van $a \in A$ genoem as

$$ab = ba, \; b = bab \text{ en } a - aba \in \mathrm{N}(A).$$

Jare later het Koliha die definisie van Drazin soos volg veralgemeen: As A ’n Banach algebra is, dan word ’n element $b \in A$ ’n veralgemeende Drazin inverse van $a \in A$ genoem as

$$ab = ba, \; b = bab \text{ en } a - aba \in \mathrm{QN}(A).$$

Vanweë die uniekheid van die Drazin (veralgemeende Drazin) inverse, laat $a^d(a^D)$ die Drazin (veralgemeende Drazin) inverse van ’n Banach algebra element a aandui, en laat $A^d(A^D)$ die versameling van Drazin (veralgemeende Drazin) inverteerbare elemente in ’n Banach algebra A aandui.

Dit is welbekend dat die funksie wat ’n inverteerbare element op sy inverse afbeeld kontinu is. In hierdie tesis voorsien ons voorwaardes waaronder die afbeeldings $a \mapsto a^d$ en $a \mapsto a^D$ kontinu op A^d en A^D, onderskeidelik, is. Uiteindelik pas ons hierdie resultate toe op die Banach algebra van begrensde lineêre operatore.

Acknowledgements

I wish to sincerely thank God for the strength and wisdom He hath given me to handle this dissertation successfully.

"... for without Me, you can do nothing " *John* 15:5

My profound gratitude goes to my supervisor Prof. Sonja Mouton for her excellent guidance and help in developing me as a mathematician. Her constant encouragement, helpful suggestions and always present constructive corrections brought about the completion of this thesis.

I would also like to address my gratitude toward the head of the Department of Mathematical Sciences, Prof. I. Rewitzky, and the other staff members in the Mathematics Division at Stellenbosch University for their assistance and advice. A special thank you to Mrs L. Adams and Mrs O. Marais whom I could always count on for a quick favour. I appreciate my fellow students in the Mathematics Division for their advice and friendship during the course of this study. Thank you, Hannes Bezuidenhout, for your willingness to proofread my work.

I am greatly indebted to the Ernst and Ethel Eriksen trust for their financial support that enabled me to complete this programme.

I must not fail to thank my family and friends for their continuous prayers and inspirational encouragement. A special thanks to Mrs Melanie Johnson and Mrs Marie Trantaal for their motherly care during the course of my studies. My appreciation also goes to Mr Gabriel and Mrs Eunice Cillie for creating a comfortable environment for me to study in throughout my master studies.

Last, but not least, I would like to address my gratitude toward my parents for their love, support and the continuous sacrifices they have made to see my dreams being accomplished in life. I appreciate you.

Dedications

To my dearest father, Rudolph Benjamin, and mother, Sylvia Benjamin, who have raised me well enough to believe that dreams can come true. You have been my biggest supporters, even though you have no clue what I am doing. I take this time to honour you.

Contents

Nomenclature

Sets and spaces

$\mathbb{N}$	the set of natural numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$K(\mathbb{C})$	the set of all compact subsets of $\mathbb{C}$
$H(\Omega)$	the algebra of all complex valued functions defined and holomorphic on an open set $\Omega \subseteq \mathbb{C}$
X^*	the dual space of a normed space X
$\mathrm{M}^{\perp}$	the set of all bounded linear functionals that vanishes on a subset M of a normed space
X/N	the set of equivalence classes of a closed subspace N of a Banach space X
$\ell^{\infty}(A)$	the algebra of all bounded sequences of elements in a Banach algebra A
$c(A)$	the algebra of all convergent sequences of elements in a Banach algebra A
$C(K)$	the algebra of all complex continuous functions on a non-empty compact set K
$M_n(\mathbb{C})$	the algebra of all $n \times n$ matrices with complex entries
$\mathfrak{L}(X)$	the algebra of all bounded linear operators on a Banach space X
$\mathrm{Null}(T)$	the null space of $T \in \mathfrak{L}(X)$
$\mathrm{R}(T)$	the range of $T \in \mathfrak{L}(X)$
$\mathrm{Comm}\ a$	the commutator algebra of an algebra element a
$\mathrm{Comm}^2\ a$	the double commutant of an algebra element a
$\mathrm{N}(A)$	the set of nilpotent elements of an algebra A
$\mathrm{QN}(A)$	the set of quasinilpotent elements of a Banach algebra A
$\mathrm{Rad}(A)$	the radical of a Banach algebra A
$\mathrm{Soc}(A)$	the socle of a Banach algebra A

A[a] the smallest closed subalgebra of a Banach algebra A containing $a, \mathbf{1}$ and all elements of the form $(a - \lambda\mathbf{1})^{-1}$ for $\lambda \notin \sigma(a)$

Operators

T^* the adjoint of $T \in \mathfrak{L}(X)$

$\tilde{T}$ the canonical operator from $X/\text{Null}(T)$ into $\overline{\text{R}(T)}$

Other symbols

$\mathbf{1}$ identity in an algebra

$||x||$ the norm of an element x

$\langle x, y\rangle$ the inner product of vector space elements x and y

$\#B$ the number of elements in a set B

$\overline{B}$ the closure of a set B

B^o the interior of a set B

$\mathcal{B}(a, r)$ the open ball with centre a and radius r

$\mathcal{B}'(a, r)$ the open ball excluding the centre a and with radius r

$\overline{\mathcal{B}}(a, r)$ the closed ball with centre a and radius r

$\text{D}(a, B)$ the distance between a metric space element a and a subset B of a metric space

$\triangle(K_1, K_2)$ Hausdorff distance between the compact sets $K_1, K_2 \subseteq \mathbb{C}$

$\text{gap}(M, N)$ the gap between closed subspaces M and N of a Banach space

$\delta(M, N)$ the supremum over all distances between u and N where $u \in M$ with $||u|| = 1$

$\text{asc}(T)$ the ascent of $T \in \mathfrak{L}(X)$

$\text{des}(T)$ the descent of $T \in \mathfrak{L}(X)$

$j(T)$ the minimum modulus of $T \in \mathfrak{L}(X)$

$\gamma(T)$ the reduced minimum modulus of $T \in \mathfrak{L}(X)$

$p \sim q$ the idempotents p and q of a Banach algebra are similar

$\sigma(a)$ the spectrum of a Banach algebra element a

$\sigma'(a)$ the set of all non-zero elements of $\sigma(a)$

$\text{iso}\ \sigma(a)$ the set of isolated spectral points of a Banach algebra element a

$\text{acc}\ \sigma(a)$ the set of accumulation spectral points of a Banach algebra element a

$\rho(a)$ the resolvent set of a Banach algebra element a

$r(a)$ the spectral radius of a Banach algebra element a

$\text{rank}(a)$ the spectral rank of a semisimple Banach algebra element a

$\mathcal{F}$	the set of all spatially finite-rank elements of the relevant semisimple Banach algebra
$\mathcal{G}$	the set of all spectrally finite-rank elements of the relevant semisimple Banach algebra
a^{-1}	the inverse of an algebra element a
A^{-1}	the set of invertible elements of an algebra A
a^g	the group inverse of an algebra element a
A^g	the set of group invertible elements of an algebra A
a^d	the Drazin inverse of an algebra element a
A^d	the set of Drazin invertible elements of an algebra A
a^D	the generalized Drazin inverse of a Banach algebra element a
A^D	the set of generalized Drazin invertible elements of a Banach algebra A
$a^{(c)}$	the core of a Drazin or generalized Drazin invertible element a
$a^{(n)}$	the nilpotent part of a Drazin invertible element a
$a^{(q)}$	the quasinilpotent part of a generalized Drazin invertible element a

Chapter 1

Introduction

The theory of generalized inverses of elements in some algebraic structure extends the concept of an inverse of an invertible element to non-invertible elements.

Let $X \in M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ denotes the Banach algebra of all $n \times n$ matrices with complex entries. In 1955 R. Penrose (see [22], Theorem 1) established the existence and uniqueness of a matrix $B \in M_n(\mathbb{C})$ satisfying

$$XBX = X, \ BXB = B, \ (XB)^T = XB \text{ and } (BX)^T = BX,$$

where X^T denotes the conjugate transpose of X. The matrix B is known as the Moore-Penrose inverse, which is a generalized inverse of the matrix X.

Since then, many authors have proposed and investigated various types of generalized inverses in a number of algebraic structures, causing the theory of generalized inverses to see a substantial growth over the past decades. This theory covers a wide range of mathematical areas such as matrix theory and operator theory, and has found applications in various areas including differential equations and Markov chains.

With some modifications of the Moore-Penrose inverse, M. P. Drazin generalized this concept in 1958 to arbitrary semigroups and associative rings. For the purpose of this thesis we will study this concept in the context of a general algebra. If A denotes an algebra, then we call an element $b \in A$ a Drazin inverse of $a \in A$ if

$$ab = ba, \ b = bab \text{ and } a^k = a^k ba,$$

for some $k \in \mathbb{N}$. If $k = 1$, then b is called a group inverse of a. Also, if $\mathrm{N}(A)$ denotes the set of all nilpotent elements of A, then the condition $a^k = a^k ba$ for some $k \in \mathbb{N}$ is equivalent to the condition $a - aba \in \mathrm{N}(A)$ (see Lemma 4.1.3). It is then clear that every nilpotent element is Drazin invertible with 0 as a Drazin inverse. We denote by A^d the set of all Drazin invertible elements of A.

The concept of a Drazin inverse was further generalized in 1996 by J. J. Koliha in [12]. Let A be a Banach algebra and let $\mathrm{QN}(A)$ denote the set of all quasinilpotent elements of A. Then we call an element $b \in A$ a generalized Drazin inverse of $a \in A$ if

$$ab = ba, \; b = bab \text{ and } a - aba \in \mathrm{QN}(A).$$

It is easy to see from the definition of a generalized Drazin inverse that every quasinilpotent element is generalized Drazin invertible with 0 as a generalized Drazin inverse. We use the notation A^D to indicate the set of all generalized Drazin invertible elements of A.

In Lemma 4.1.5 and Corollary 5.1.8, respectively, we establish the uniqueness of the Drazin inverse and the generalized Drazin inverse, provided they exist. Let a^d and a^D denote, respectively, the Drazin inverse and the generalized Drazin inverse of an element a in a Banach algebra A. It is well-known that inversion is continuous on the set of invertible elements in A (see Theorem 2.7.1). Natural questions are now whether the maps $a \mapsto a^d$ and $a \mapsto a^D$ are continuous on A^d and A^D, respectively. The answer is no in general (see Example 6.2.1). The purpose of this thesis is then to study the continuity properties of these maps.

We give a short introduction to the theory studied in each chapter. The purpose of Chapter 2 is to present definitions and results that will be used throughout this thesis.

In Chapter 3 we introduce and discuss the concept of a group inverse in an algebra. Let A^g denote the set of group invertible elements of an algebra A. As mentioned above, this particular generalized inverse is a special case of the Drazin inverse, so that the inclusion $A^g \subseteq A^d$ holds in general.

The aim of Section 3.1 is to investigate basic properties like the existence and uniqueness of the group inverse. In Proposition 3.1.4 we establish the uniqueness of the group inverse, provided it exists. Let a^g denote the group inverse of a. The terminology comes from the fact that $\{a, a^g\}$ generates a group with identity aa^g. Since non-zero nilpotent elements (which are Drazin invertible) are not group invertible (Lemma 3.1.5), it follows that the inclusion $A^g \subseteq A^d$ is strict in general. Necessary and sufficient conditions for the existence of a group inverse are presented in Proposition 3.1.6.

In Section 3.2 we describe the spectrum of a group invertible element in a Banach algebra. Roch and Silbermann showed, in [25], that the condition $0 \notin \operatorname{acc} \sigma(a)$ is satisfied if a is a group invertible element (see Lemma 3.2.1). In the same paper they found that, in semisimple commutative Banach algebras, the spectral condition is sufficient (Proposition 3.2.4), yielding yet another necessary and sufficient condition for the existence of a group inverse. We will, however, give a different proof for this result than that of Roch and Silbermann in [25].

In Chapter 4 we introduce and study the concept of a Drazin invertible element in an algebra. Most of the work done in this chapter comes from the paper by Roch and Silbermann (see [25]).

In Section 4.1 various results that were obtained for group inverses in Chapter 3 will be extended to the general case of Drazin invertibility. We already mentioned the uniqueness of the Drazin inverse above (recalling Lemma 4.1.5). In Lemma 4.1.6 we present a relation between the Drazin inverse and the group inverse. Using Lemma 4.1.6 we obtain an analogue of Proposition 3.1.6 for Drazin inverses in Proposition 4.1.9. This result characterizes the existence of a Drazin inverse in terms of idempotents.

Section 4.2 is aimed at discussing the spectrum of a Drazin invertible element in a Banach algebra. In Lemma 4.2.1 we present an analogue of Proposition 3.2.4 for Drazin inverses. This result implies that, in a semisimple commutative Banach algebra A, Drazin invertiblity is equivalent to group invertibility, that is, $A^g = A^d$ (see Corollary 4.2.2).

In Chapter 5 we present and study a generalization of the Drazin inverse introduced in Chapter 4, called the generalized Drazin inverse. This concept was introduced and investigated by Koliha in [12]. Most of his work done in this paper will be presented in this chapter. Let us mention that this generalization will provide the cornerstone for this thesis, while the group inverse and the Drazin inverse are investigated to a lesser degree. In our discussion of Chapter 5 we will always work in a Banach algebra A.

The purpose of Section 5.1 is to address logical issues such as the existence and uniqueness of the generalized Drazin inverse in a Banach algebra A. By the definitions of the Drazin inverse and the generalized Drazin inverse, the inclusion $A^d \subseteq A^D$ holds. In Example 5.1.3 we show that the inclusion may in general be strict. In Proposition 5.1.4 we present an analogue of Proposition 4.1.9 for generalized Drazin inverses, which gives necessary and sufficient conditions for the existence of a generalized Drazin inverse. A key result, due to Koliha, is Theorem 5.1.6. Let us mention that this result is particularly useful in developing the theory of generalized Drazin inverses. It says that, for $a \in A$, the property $0 \notin \operatorname{acc} \sigma(a)$ holds if and only if there exists an idempotent $p \in A$ satisfying the conditions

$$ap = pa,\ a + p \text{ is invertible and } ap \in \mathrm{QN}(A).$$

If we combine this result with Proposition 5.1.4, we obtain the following conclusion: $a \in A^D$ if and only if $0 \notin \operatorname{acc} \sigma(a)$. Theorem 5.1.6, together with the fact that the idempotent p is actually unique, leads to the establishment of the uniqueness of the generalized Drazin inverse (Corollary 5.1.8). Moreover, this result, under the condition $0 \in \operatorname{iso} \sigma(a)$, enables us to partition the set A^D into elements which are Drazin invertible (Corollary 5.1.10) and elements which are generalized Drazin invertible but not Drazin invertible (Corollary 5.1.11). Following R. E. Harte, an element $a \in A$ is said to be quasipolar if there exists an idempotent $q \in A$ such that

$$aq = qa,\ q \in Aa \cap aA \text{ and } a(\mathbf{1} - q) \in \mathrm{QN}(A).$$

Harte then established the following spectral characterization of quasipolar elements (see Theorem 5.1.13): An element $a \in A$ is quasipolar if and only if $0 \notin \text{acc}\ \sigma(a)$. This result, together with the conclusion above, shows that the quasipolar elements coincide with the generalized Drazin invertible elements (Theorem 5.1.16). It also follows from Theorem 5.1.16 that, if A is a semisimple commutative Banach algebra, then the sets A^g, A^d and A^D are identical (see Corollary 5.1.18).

In Section 5.2 we discuss a number of algebraic properties of generalized Drazin invertible elements.

Section 5.3 deals with the decomposition of a generalized Drazin invertible element. Our main result in this chapter is Theorem 5.3.1, due to Koliha. This result allows us to decompose a generalized Drazin (Drazin) invertible element a as the sum of a group invertible element x and a quasinilpotent (nilpotent) element y such that $xy = 0 = yx$ (Corollary 5.3.3; Corollary 5.3.4). It should be noted that Corollary 5.3.4 is a generalization of the well-known core-nilpotent decomposition of a square matrix ([4], Theorem 11, p.169). C. F. King obtained a similar result in [11] for bounded linear operators.

In Section 5.4 we discuss the spectrum of the generalized Drazin inverse of an element in A^D. In [12] Koliha showed that, if $0 \in \text{iso}\ \sigma(a)$, then the non-zero spectrum of a^D consists of the reciprocals of the non-zero spectral points of a (Theorem 5.4.1).

In Chapter 6 we present the continuity properties of the Drazin inverse and the generalized Drazin inverse of elements in a general Banach algebra A.

In Section 6.1 we introduce and investigate the notion of inverse closedness of subalgebras. In general, a subalgebra B of A which is inverse closed with respect to invertibility is not necessarily inverse closed with respect to generalized invertibility (Example 6.1.1). In Proposition 6.1.2 and Corollary 6.1.4 we show that if B is closed and contains the identity, then B being inverse closed with respect to invertibility implies that B is inverse closed with respect to both group invertibility and Drazin invertibility. These results will be used in Section 6.2.

In Section 6.2 we present characterizations of continuity of group, Drazin and generalized Drazin inversion in Banach algebras. We start by examining analogies between the continuity of inversion (Section 2.7) and the continuity of group, Drazin and generalized Drazin inversion. First, we demonstrate (see Example 6.2.1) that group, Drazin and generalized Drazin inversion is not in general continuous on A^g, A^d and A^D, respectively (unlike inversion on A^{-1}). In Proposition 6.2.3 and Corollary 6.2.4, respectively, we prove that an analogue of Lemma 2.7.2 is, however, available for group inverses and Drazin inverses. In [13] Koliha and V. Rakočević showed that Lemma 2.7.2, unfortunately, does not hold when replacing invertibility by generalized Drazin invertibility (see Example 6.2.6). Finally, we found that an analogue of Lemma 2.7.3 is not possible for group, Drazin and generalized Drazin inverses (Example 6.2.7; Remark 6.2.8), since the sets A^g, A^d and A^D are not open in general (Example

6.2.9; Remark 6.2.10).

In Lemma 6.2.12 we describe the convergence of the group inverses of a convergent sequence of group invertible elements in terms of the convergence of their group idempotents. In Theorem 6.2.13, which is our main result in this section, we show that an analogue of Lemma 6.2.12 is possible for generalized Drazin inverses. A number of continuity properties of generalized Drazin inverses of elements in a Banach algebra are presented in Theorem 6.2.13, some of which Rakočević already proved for Drazin invertible elements (see [24], Theorem 4.1). From Theorem 6.2.13 we also have that an analogue of Lemma 2.7.2 is available for generalized Drazin inverses, under the assumption that the limit of the convergent sequence of generalized Drazin invertible elements is also generalized Drazin invertible.

In Section 6.3 we present criteria, using the concept of the spectral rank, for continuity of group and Drazin inversion in the special case of the socle elements of semisimple Banach algebras, as was done by R. M. Brits, L. Lindeboom and H. Raubenheimer in [7]. In particular, we mention the important role played by the equivalences of (a) and (d) and of (a) and (f) in Theorem 6.2.13 in order to obtain these criteria. In our discussion of Section 6.3 we will always work in a semisimple Banach algebra A. In Theorem 6.3.1 we show that the socle elements of A are a special class of Drazin invertible elements. We continue by providing conditions which ensure that $a_n^d \to a^d$ as $n \to \infty$, given that $a_n \to a$ in $\mathrm{Soc}(A)$ as $n \to \infty$. In Lemma 6.3.4 we first restrict ourselves to maximal finite-rank elements. We then extend this result to arbitrary group invertible elements in $\mathrm{Soc}(A)$ (see Theorem 6.3.5), from which we generalize the result further to arbitrary socle elements in A (Corollary 6.3.6). In ([8], Theorem 10.7.1) S. L. Campbell and C. D. Meyer characterized the continuity of the Drazin inverse of a square matrix in terms of the ranks of the core matrices. It should be noted that Corollary 6.3.6 generalizes Campbell and Meyer's continuity result to arbitary socle elements in semisimple Banach algebras.

In Chapter 7 we discuss the theory of the generalized Drazin inverse of a bounded linear operator and study the continuity of the generalized Drazin inverse in $\mathfrak{L}(X)$, where $\mathfrak{L}(X)$ denotes the algebra of bounded linear operators on a Banach space X. Authors such as Campbell and Meyer studied the continuity of the Drazin inverse of a square matrix (see [8]), while Rakočević investigated the continuity properties of the Drazin inverse of a bounded linear operator (see, for instance, [24]). The work done in this chapter comes from the paper [13] by Koliha and Rakočević.

In Section 7.1 we present various important consequences of the core-quasinilpotent decomposition of a bounded linear operator (Theorem 7.1.1). In particular, we find that the core operator of an element $A \in \mathfrak{L}(X)^D$ is a closed range operator (Corollary 7.1.6).

The results presented in Section 7.2 are key results in our approach to the study of the continuity of the generalized Drazin inverse in $\mathfrak{L}(X)$. These results present properties that are satisfied by the gap function and involve idempotents in $\mathfrak{L}(X)$. An important result in this section is Lemma 7.2.4, which describes the convergence of idempotents in $\mathfrak{L}(X)$ in terms of the convergence of the gaps of null spaces and ranges.

Finally, in Section 7.3 we specialize the continuity results obtained for the generalized Drazin inverse in the Banach algebra setting to the bounded linear operator case (Theorem 7.3.3) and formulate, with the help of notions like the gap and the reduced minimum modulus, more characterizations of continuity of generalized Drazin inversion of bounded linear operators.

Chapter 2

Preliminaries

This chapter contains definitions and results that will be relevant in the rest of the thesis. The proofs of results that can be found in standard texts on Banach algebra theory will be omitted, while the proofs of results from papers that may not be readily available will be given. There are instances where a fact that is of particular importance in a result has simply been stated by the authors in their paper. The complete proofs of such results are also supplied below. It is expected that the reader has a good comprehension of algebra, complex analysis, functional analysis and topology.

2.1 Banach algebra theory

Definition 2.1.1 (Algebra) *([15], p.394) An algebra is a vector space A over a field K such that for each ordered pair of elements $x, y \in A$, a unique product $xy \in A$ is defined satisfying the properties*

- $x(yz) = (xy)z$
- $(x + y)z = xz + yz$
- $x(y + z) = xy + xz$
- $\lambda(xy) = (\lambda x)y = x(\lambda y)$
- *there exists an element* $\mathbf{1} \in A$ *such that* $\mathbf{1}x = x = x\mathbf{1}$

for all x, y, z in A, $\lambda \in K$ and where $\mathbf{1}$ denotes the identity of A.

An algebra A is said to be *commutative* if $xy = yx$ for all $x, y \in A$. If $K = \mathbb{R}$ in Definition 2.1.1 then A is called a *real algebra*, whereas if $K = \mathbb{C}$ then A is said to be a *complex algebra*. From this point on we use the word "algebra" to mean "complex algebra."

Definition 2.1.2 (Inner product space) *([15], Definition 3.1.1) An inner product space is a vector space A with an inner product defined on it. Here, an inner product on a complex vector space A is a mapping from $A \times A$ into $\mathbb{C}$ whose value is denoted by $\langle x, y \rangle$ for each ordered pair of elements $x, y \in A$, and which has the properties*

- $\langle x, x \rangle \geq 0$
- $\langle x, x \rangle = 0 \Leftrightarrow x = 0$
- $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- $\langle x, y \rangle = \overline{\langle y, x \rangle}$

for all x, y, z in A, $\alpha \in \mathbb{C}$ and where $\overline{x}$ denotes the complex conjugate of the element x.

A complete inner product space is called a *Hilbert space.*

Definition 2.1.3 (Normed space) *([15], Definition 2.2.1) A normed space is a vector space A with a norm defined on it. Here, a norm on a complex vector space A is a real-valued function on A whose value is denoted by $|| \cdot ||$ and which has the properties*

- $||x|| \geq 0$
- $||x|| = 0 \Leftrightarrow x = 0$
- $||\alpha x|| = |\alpha| ||x||$
- $||x + y|| \leq ||x|| + ||y||$

for all x, y in A and $\alpha \in \mathbb{C}$.

A complete normed space is called a *Banach space.*

Definition 2.1.4 (Banach algebra) *([2], Definition p.30) If an algebra A is a Banach space for $|| \cdot ||$, satisfies the norm inequality $||xy|| \leq ||x|| ||y||$ for all $x, y \in A$ and $||\mathbf{1}|| = 1$, then we say that A is a Banach algebra.*

The following are some examples of Banach algebras:

- ([2] Example 1, p.31). Let K be a non-empty compact set. Then *C(K)*, the vector space of all complex continuous functions on K, with the supremum norm and with product xy defined by

$$(xy)(t) = x(t)y(t)$$

for all $x, y \in C(K)$ and $t \in K$, is a Banach algebra. This is also an example of a commutative Banach algebra.

- ([2], Example 5, p.32). Let X be a complex Banach space with dim $X \geq 1$. Then $\mathfrak{L}(X)$, the vector space of all bounded linear operators on X, with the usual operator norm and with product the composition of operators; that is, if $S, T \in \mathfrak{L}(X)$ and $x \in X$, then

$$(ST)x = S(Tx),$$

is a Banach algebra

- ([2], p.32) The algebra of all $n \times n$ matrices with complex entries $M_n(\mathbb{C})$ is a Banach algebra with norm

$$||A|| = \sup\left\{\sum_{j=1}^{n} |a_{ij}| : i = 1, \ldots, n\right\},$$

where $A = (a_{ij})$.

We need the following definition and result about metric spaces.

Definition 2.1.5 (Distance) *Let (A, d) be a metric space, $a \in A$ and $B \subseteq A$. The distance between a and B, denoted by $\mathrm{D}(a, B)$, is defined by*

$$\mathrm{D}(a, B) = \begin{cases} \inf\{d(a, b) : b \in B\} & \text{if } B \neq \emptyset \\ \infty & \text{if } B = \emptyset. \end{cases}$$

The notations $\overline{B}$ and B^o indicate the closure and interior of the set B, respectively.

Theorem 2.1.6 (Baire's Category Theorem) *([17], Theorem 1.5.4) Let A be a non-empty complete metric space. If for each $k \in \mathbb{N}$ there exists a subset U_k of A such that $\overline{U_k^o} = A$, then $\overline{\cap_{k=1}^{\infty} U_k^o} = A$, and hence $\overline{\cap_{k=1}^{\infty} U_k} = A$.*

Definition 2.1.7 (Invertible elements) *([15], p.396) Let A be an algebra. An element $a \in A$ is said to be invertible if there exists an element $a^{-1} \in A$ such that $aa^{-1} = a^{-1}a = \mathbf{1}$.*

The element a^{-1} in the definition above is called the *inverse* of a. We write A^{-1} for the set of invertible elements of an algebra A. Note that A^{-1} forms a group with respect to multiplication, containing the identity.

Lemma 2.1.8 (N. Jacobson) *([2], Lemma 3.1.2) Let A be an algebra, $a, x \in A$ and $\lambda \neq 0$. Then $\lambda\mathbf{1} - ax \in A^{-1}$ if and only if $\lambda\mathbf{1} - xa \in A^{-1}$.*

Theorem 2.1.9 (Neumann series) *([2], Theorem 3.2.1) Let A be a Banach algebra. If $a \in A$ satisfies $||a|| < 1$, then $\mathbf{1} - a \in A^{-1}$ and we have that*

$$(\mathbf{1} - a)^{-1} = \sum_{k=0}^{\infty} a^k.$$

Definition 2.1.10 (Nilpotent) *Let A be an algebra. An element $a \in A$ is nilpotent if there exists a positive integer n such that $a^n = 0$.*

The set of all nilpotent elements of an algebra A will be denoted by $\mathrm{N}(A)$.

Let a be an element of an algebra. The *commutator algebra* of the element a, being the set of all elements which commute with a, is denoted by $\mathrm{Comm}\, a$. The *double commutant* of a, denoted by $\mathrm{Comm}^2\, a$, is the set of all elements that commute with every element in $\mathrm{Comm}\, a$. It can be easily verified that, if a is a Banach algebra element, then the sets $\mathrm{Comm}\, a$ and $\mathrm{Comm}^2\, a$ are closed.

The following two lemmas will be required in Chapter 4. We will need Lemma 2.1.11 in the proof of Proposition 4.1.9, while Lemma 2.1.12 will be used in the proof of Lemma 4.1.13.

Lemma 2.1.11 *([25], p.202) Let A be an algebra. If $r \in \mathrm{N}(A) \cap \mathrm{Comm}\, c$, then c is invertible if and only if $c + r$ is invertible.*

Proof:
Let $r \in \mathrm{Comm}\, c$ with $r^k = 0$, for some $k \in \mathbb{N}$.

Suppose that c is invertible. Observe that $r \in \mathrm{Comm}\, c$ implies that $r \in \mathrm{Comm}\, c^{-1}$. We also have that

$$(\mathbf{1} - c^{-1}r + c^{-2}r^2 - \cdots + (-1)^{k-1}c^{-(k-1)}r^{k-1})(\mathbf{1} + c^{-1}r) = \mathbf{1} + (-1)^{k-1}c^{-k}r^k = \mathbf{1}.$$

Since $r \in \mathrm{Comm}\, c^{-1}$, it follows that

$$\mathbf{1} - c^{-1}r + c^{-2}r^2 - \cdots + (-1)^{k-1}c^{-(k-1)}r^{k-1} \in \mathrm{Comm}\, (\mathbf{1} + c^{-1}r).$$

Hence $\mathbf{1} + c^{-1}r$ is invertible, so that $c + r = c(\mathbf{1} + c^{-1}r)$ is invertible.

Conversely, suppose that $c + r$ is invertible. Since r is nilpotent, $-r$ is also nilpotent, and hence $-r \in \mathrm{N}(A) \cap \mathrm{Comm}\, c$. By the previous reasoning we have that $c = c + r + (-r)$ is invertible. This completes the proof. ■

Lemma 2.1.12 *Let A be an algebra. If $a, b \in \mathrm{N}(A)$ and $b \in \mathrm{Comm}\, a$, then $a + b \in \mathrm{N}(A)$.*

Proof:
Suppose that $a \in \mathrm{N}(A)$ and that $b \in \mathrm{N}(A) \cap \mathrm{Comm}\, a$. Let k_1 and k_2 be the smallest positive integers such that $a^{k_1} = 0$ and $b^{k_2} = 0$ and suppose that $k_1 \leq k_2$. By the binomial theorem we have that

$$(a + b)^{k_1+k_2} = \sum_{n=0}^{k_1+k_2} \binom{k_1 + k_2}{n} a^n b^{k_1+k_2-n}.$$

If $n < k_1$, then $k_1 + k_2 - n > k_1 + k_2 - k_1 = k_2$, so that $b^{k_1+k_2-n} = 0$. If $n \geq k_1$, then $a^n = 0$. Hence $(a+b)^{k_1+k_2} = 0$, so that $a + b \in \mathrm{N}(A)$. ■

Definition 2.1.13 (Idempotent) *Let A be an algebra. An element $p \in A$ is called an idempotent if p satisfies $p^2 = p$.*

Definition 2.1.14 (Similarity of idempotents) *([25], p.215) Let A be a Banach algebra and p and q idempotents in A. If there exists an invertible element $c \in A$ satisfying $q = c^{-1}pc$, then p and q are said to be similar and we write $p \sim q$.*

The following lemma will be needed in the proof of Theorem 6.3.5. It gives a sufficient condition for the existence of similar idempotents.

Lemma 2.1.15 *([25], Lemma 12) Let A be a Banach algebra and p and q idempotents in A such that $||p - q|| < ||2p - \mathbf{1}||^{-1}$. Then $p \sim q$.*

Proof:
Suppose that p and q are idempotents in A satisfying $||p - q|| < ||2p - \mathbf{1}||^{-1}$. Observe that the element $2p-\mathbf{1}$ is invertible with inverse itself. By assumption we have that

$$||(2p - \mathbf{1})^{-1}(q - p)|| \leq ||(2p - \mathbf{1})^{-1}|| \, ||q - p|| < ||2p - \mathbf{1}|| \, ||2p - \mathbf{1}||^{-1} = 1,$$

and hence $\mathbf{1}+(2p-\mathbf{1})^{-1}(q-p)$ is invertible by Theorem 2.1.9. Let $c := p+q-\mathbf{1}$. It then follows that

$$c = (2p - \mathbf{1}) + (q - p) = (2p - \mathbf{1})[\mathbf{1} + (2p - \mathbf{1})^{-1}(q - p)]$$

is also invertible. Now,

$$pc = p(p + q - \mathbf{1}) = p + pq - p = pq = pq + q - q = (p + q - \mathbf{1})q = cq,$$

that is, $q = c^{-1}pc$; hence $p \sim q$. ■

Next, we formulate several definitions relating to the concept of an ideal of a Banach algebra.

Definition 2.1.16 *Let A be a Banach algebra. A subset J of A is said to be a left (right) multiplicative ideal of A if*

- $J \neq \emptyset$
- $AJ \subseteq J$ $(JA \subseteq J)$

Definition 2.1.17 *Let A be a Banach algebra. A vector space $J \subseteq A$ is said to be a left (right) ideal of A if*

- $J \neq \emptyset$

- $AJ \subseteq J$ $(JA \subseteq J)$

If a vector space J of A satisfies both the conditions of a left and a right ideal, then J is called a *two-sided ideal* of A.

Definition 2.1.18 (Minimal ideals) *([5], Definition 1, p.154) Let A be a Banach algebra. A non-zero left (right, two-sided) ideal M of A is said to be a minimal left (right, two-sided) ideal of A if there exists no left (right, two-sided) ideal I of A satisfying $\{0\} \subsetneq I \subsetneq M$.*

2.2 Spectral theory in Banach algebras

Definition 2.2.1 (Spectrum) *([2], Definition p.36) Let A be a Banach algebra. The spectrum of an element $a \in A$, denoted by $\sigma_A(a)$, is defined as follows:*

$$\sigma_A(a) := \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \notin A^{-1}\}$$

Note that we will only write $\sigma(a)$ if the Banach algebra under discussion is clear from the context. By $\sigma'(a)$ we denote the set of all non-zero elements of $\sigma(a)$. Let us remark that the equality $\sigma'(ax) = \sigma'(xa)$, for Banach algebra elements a and x, is a direct consequence of Lemma 2.1.8.

Examples of the spectrum of a Banach algebra element include:

- If $f \in C[a, b]$, where $[a, b] \subseteq \mathbb{R}$, then $\sigma(f) = f[a, b]$.
- If $A \in M_n(\mathbb{C})$, then $\sigma(A) = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } A\}$.

The set of all isolated spectral points of a Banach algebra element a will be denoted by iso $\sigma(a)$, while the set of all accumulation points of the spectrum of a will be denoted by acc $\sigma(a)$. The notation $\rho(a)$ will be used to denote the complement of $\sigma(a)$. We call $\rho(a)$ the *resolvent set of a*.

Definition 2.2.2 (Spectral radius) *([2], Definition p.36) Let A be a Banach algebra and $a \in A$. The spectral radius $r_A(a)$ is defined as follows:*

$$r_A(a) := \sup\{|\lambda| : \lambda \in \sigma_A(a)\}$$

It suffices to write $r(a)$ if the Banach algebra being discussed is clear from the context.

Theorem 2.2.3 (I.M. Gelfand) *([2], Theorem 3.2.8) Let A be a Banach algebra and $a \in A$. Then*
(i) $\lambda \mapsto (\lambda\mathbf{1} - a)^{-1}$ is analytic on $\rho(a)$,
(ii) $\sigma(a)$ is compact and non-empty,
(iii) $r(a) = \lim_{n\to\infty} ||a^n||^{\frac{1}{n}}$.

Statement (ii) in Theorem 2.2.3 implies that the spectrum is a closed and bounded subset of $\mathbb{C}$, while statement (iii) implies that $r(a) \leq ||a||$, for a Banach algebra element a.

Definition 2.2.4 (Character) *([2], p.69) Let A be a Banach algebra. A linear map $\chi : A \to \mathbb{C}$ is called a character of A if it has the properties*

- $\chi(ab) = \chi(a)\chi(b)$
- $\chi(\mathbf{1}) = 1$

for all $a, b \in A$.

Theorem 2.2.5 (I.M. Gelfand) *([2], Theorem 4.1.2) Let A be a commutative Banach algebra and $a \in A$. Then $\sigma(a) = \{\chi(a) : \chi$ is a character of $A\}$.*

Corollary 2.2.6 *([2], Corollary 3.2.10) Let A be a Banach algebra and $a, b \in A$. If $ab = ba$, then $\sigma(a + b) \subseteq \sigma(a) + \sigma(b)$ and $\sigma(ab) \subseteq \sigma(a)\sigma(b)$. Hence $r(a + b) \leq r(a) + r(b)$ and $r(ab) \leq r(a)r(b)$.*

Definition 2.2.7 (Quasinilpotent) *([2], p.36) Let A be a Banach algebra. An element $a \in A$ is quasinilpotent if $\sigma(a) = \{0\}$.*

By $\mathrm{QN}(A)$ we indicate the set of all quasinilpotent elements of a Banach algebra A. By using the spectral mapping theorem for polynomials, it can be easily shown that $\mathrm{N}(A) \subseteq \mathrm{QN}(A)$.

The *radical* of a Banach algebra A, denoted by $\mathrm{Rad}(A)$, is defined by

$$\mathrm{Rad}(A) = \{a \in A : za \in \mathrm{QN}(A), \text{ for all } z \in A\}.$$

It is well-known that $\mathrm{Rad}(A)$ is a two-sided ideal of a Banach algebra A. If $\mathrm{Rad}(A) = \{0\}$, then A is called *semisimple*.

Examples of semisimple Banach algebras are $M_n(\mathbb{C})$, $\mathfrak{L}(X)$ and $C[a, b]$, where $[a, b] \subseteq \mathbb{R}$. If A is a semisimple Banach algebra and $p \in A$ is an idempotent, then pAp is another example of a semisimple Banach algebra, with identity p.

The following well-known result about semisimple Banach algebras will be required in the proof of Theorem 6.3.5. It states that every semisimple algebra that is finite-dimensional over $\mathbb{C}$ is a direct sum of matrix algebras over $\mathbb{C}$.

Theorem 2.2.8 (J. H. M. Wedderburn-E. Artin) *([2], Theorem 2.1.2) Let A be a semisimple finite dimensional algebra over $\mathbb{C}$. Then there exist integers $n_1, \ldots, n_k \geq 1$ such that $A \simeq M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$.*

The following result will be required in Section 6.3.

Lemma 2.2.9 *([1], Lemma 6, p.4) Let A be a Banach algebra, $a \in A$ and $p \in A$ an idempotent. Then $\sigma'_A(pap) = \sigma'_{pAp}(pap)$.*

Lemma 2.2.9 implies that if $b \in pAp$, where p is an idempotent of a Banach algebra A, then we may refer unambiguously to $\sigma'(b)$.

It is clear from the definition of the radical of a Banach algebra A that $\mathrm{Rad}(A) \subseteq \mathrm{QN}(A)$. However, the following lemma, which follows from Theorem 2.2.3(ii) and Corollary 2.2.6, shows that the inclusion can be replaced by an equality sign if the Banach algebra is commutative. This result will be used in the proof of Proposition 3.2.4.

Lemma 2.2.10 *([2], p.71) Let A be a commutative Banach algebra. Then* $\mathrm{Rad}(A) = \mathrm{QN}(A)$.

We will need the following two definitions in Chapter 6.

Definition 2.2.11 *([25], p.206) Let A be a Banach algebra and $a \in A$. The smallest closed subalgebra of A which contains a, $\mathbf{1}$ and all elements of the form $(a - \lambda\mathbf{1})^{-1}$ for $\lambda \notin \sigma(a)$, is denoted by $A[a]$.*

The algebra $A[a]$ will play an important role in the proofs of Proposition 6.1.2 and Corollary 6.1.4.

Definition 2.2.12 (Inverse closedness) *([25], p.205) Let A be a Banach algebra. A subalgebra B of A is said to be inverse closed if B is inverse closed with respect to invertibility, that is, if B contains the inverses of all its invertible elements.*

If a is an element of a Banach algebra A, then Comm a is an example of an inverse closed subalgebra of A.

The following information about the algebra $A[a]$ will be required in Chapter 6.

Lemma 2.2.13 *([25], Lemma 7) The algebra $A[a]$ is the smallest closed and inverse closed subalgebra of A which contains $\mathbf{1}$ and a.*

2.3 Holomorphic Functional Calculus

The algebra of all complex-valued functions defined and holomorphic on an open set $\Omega \subseteq \mathbb{C}$ will be denoted by $H(\Omega)$.

Proposition 2.3.1 *([2], p.43) Let A be a Banach algebra and $a \in A$. If Ω is an open set containing $\sigma(a)$ and Γ is a smooth contour included in Ω that surrounds $\sigma(a)$, then the function $f \mapsto f(a) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda\mathbf{1} - a)^{-1}d\lambda$ from $H(\Omega)$ into A is well-defined.*

Theorem 2.3.2 (Holomorphic Functional Calculus (HFC)) *([2], Theorem 3.3.3, [21], Theorem 3.3.7) Let A be a Banach algebra and $a \in A$. Suppose that Ω is an open set containing $\sigma(a)$ and that Γ is a smooth contour* *included in Ω and surrounding $\sigma(a)$. Then the function defined in Proposition 2.3.1 has the following properties:*

(1) $(f_1 + f_2)(a) = f_1(a) + f_2(a)$.
(2) $(f_1 f_2)(a) = f_1(a) f_2(a)$.
(3) $1(a) = \mathbf{1}$ *and* $I(a) = a$, *where I is the identity function on $\mathbb{C}$.*
(4) If (f_n) converges to f uniformly on all compact subsets of Ω, then $f(a) = \lim_{n\to\infty} f_n(a)$.
(5) $\sigma(f(a)) = f(\sigma(a))$.
(6) If $\lim_{n\to\infty} a_n = a$, *then* $\lim_{n\to\infty} f(a_n) = f(a)$.

Number (5) above is called the *spectral mapping theorem for analytic functions,* which we will only refer to as the spectral mapping theorem.

Theorem 2.3.3 (Spectral idempotent) *([2], Theorem 3.3.4) Let A be a Banach algebra. Suppose that $a \in A$ has a disconnected spectrum. Let U_0 and U_1 be two disjoint open sets such that*

$$\sigma(a) \subseteq U_0 \cup U_1, \sigma(a) \cap U_0 \neq \emptyset \text{ and } \sigma(a) \cap U_1 \neq \emptyset.$$

Then there exists a non-trivial idempotent p commuting with a such that $\sigma(pa) = (\sigma(a) \cap U_1) \cup \{0\}$ and $\sigma(a - pa) = (\sigma(a) \cap U_0) \cup \{0\}$. Moreover, $p = f(a)$, where

$$f(\lambda) = \begin{cases} 0 & \text{if } \lambda \in U_0 \\ 1 & \text{if } \lambda \in U_1. \end{cases}$$

We call p in Theorem 2.3.3 the *spectral idempotent* of a.

If $\sigma(a) \cap U_1 = \{\lambda_0\}$ in Theorem 2.3.3, that is $\lambda_0 \in \operatorname{iso} \sigma(a)$, then the *spectral idempotent of a corresponding to λ_0* is given by

$$p = \frac{1}{2\pi i} \int_\Gamma (\lambda \mathbf{1} - a)^{-1} d\lambda$$

where Γ is a circle centred at λ_0, separating λ_0 from the remaining spectrum of a.

2.4 Continuity of the spectrum function

The notation $K(\mathbb{C})$ will be used to denote the set of all compact subsets of $\mathbb{C}$. We introduce the following metric on $K(\mathbb{C})$ in order to measure the continuity of the spectrum function.

Definition 2.4.1 (Hausdorff distance) *([2], p.48) Let $K_1, K_2 \subseteq \mathbb{C}$ be compact. The distance on $K(\mathbb{C})$ is defined by*

$$\triangle(K_1, K_2) = \max(\sup\{\mathrm{D}(z, K_1) : z \in K_2\}, \sup\{\mathrm{D}(z, K_2) : z \in K_1\}).$$

Definition 2.4.2 (Continuity of spectrum) *([2], p.48) Let A be a Banach algebra and $x \in A$. The function $x \mapsto \sigma(x)$ is continuous at $a \in A$ if, for every $\epsilon > 0$, there exists a number $\delta > 0$ such that $\triangle(\sigma(x), \sigma(a)) < \epsilon$ whenever $||x - a|| < \delta$.*

Theorem 2.4.3 (J. D. Newburgh) *([2], Theorem 3.4.4) Let A be a Banach algebra and $a \in A$. Suppose that U and V are disjoint open sets such that $\sigma(a) \subseteq U \cup V$ and $\sigma(a) \cap U \neq \emptyset$. Then there exists an $r > 0$ such that $||a - x|| < r$ implies that $\sigma(x) \cap U \neq \emptyset$.*

We denote by $\#B$ the number of elements in the set B. The following result will be required in the proof of Lemma 6.3.4.

Corollary 2.4.4 *Let A be a Banach algebra and (a_n) a convergent sequence in A with limit a. Suppose that a and all a_n have finite spectrum and that there exists $n_0 \in \mathbb{N}$ such that $\#\sigma'(a_n) = \#\sigma'(a)$ for all $n \geq n_0$. Then* $\inf\{\mathrm{D}(0, \sigma'(a_n)) : n \in \mathbb{N}\} > 0$.

Proof:
Let $a_n \to a$ as $n \to \infty$ and suppose that a and all a_n have finite spectrum. Assume also that there exists $n_0 \in \mathbb{N}$ such that $\#\sigma'(a_n) = \#\sigma'(a)$ for all $n \geq n_0$.

Suppose that $\sigma'(a) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ for some $k \in \mathbb{N}$. Let $t = \min\{|\lambda_i - \lambda_j| : i \neq j\}$ and $s = \min\left\{\frac{\mathrm{D}(0,\sigma'(a))}{2}, \frac{t}{2}\right\}$. For $i = 1, 2, \ldots, k$, the open balls $\mathcal{B}(\lambda_i, s)$ are disjoint and does not contain 0. By Theorem 2.4.3 we can find an $r_i > 0$ such that $||a - x|| < r_i$ implies that $\sigma(x) \cap \mathcal{B}(\lambda_i, s) \neq \emptyset$ for each i. Hence, if $r = \min\{r_1, \ldots, r_k\}$, then $||a - x|| < r$ implies that $\sigma(x) \cap \mathcal{B}(\lambda_i, s) \neq \emptyset$ for all $i \in \{1, \ldots, k\}$. Since $a_n \to a$ as $n \to \infty$, there exists $n_1 \in \mathbb{N}$ such that $||a - a_n|| < r$ for all $n \geq n_1$, and hence $\sigma(a_n) \cap \mathcal{B}(\lambda_i, s) \neq \emptyset$ for all $i = 1, \ldots, k$ and $n \geq n_1$. Let $N = \max\{n_0, n_1\}$. Then $\#\sigma'(a_n) = \#\sigma'(a)$ and $\sigma(a_n) \cap \mathcal{B}(\lambda_i, s) \neq \emptyset$ for all $n \geq N$ and $i = 1, \ldots, k$. Hence, each of these open balls contains a point of $\sigma'(a_n)$ for all $n \geq N$. Suppose that $\mathrm{D}(0, \sigma'(a)) = |\lambda_p|$, for some $p \in \{1, \ldots, k\}$. For $n \geq N$, suppose that $\mathrm{D}(0, \sigma'(a_n)) = |\lambda_l^{(n)}|$, for some $l \in \{1, \ldots, k\}$. By the choice of our open balls we must have that the spectral value $\lambda_l^{(n)}$ is inside $\mathcal{B}(\lambda_p, s)$, and hence $\inf\{\mathrm{D}(0, \sigma'(a_n)) : n \geq N\} > 0$, so that $\inf\{\mathrm{D}(0, \sigma'(a_n)) : n \in \mathbb{N}\} > 0$. ■

A set $K \subseteq \mathbb{C}$ is said to be *totally disconnected* if the only connected sets in K are the one-point sets. It is obvious that every finite subset of $\mathbb{C}$ is a totally disconnected set.

Corollary 2.4.5 (J. D. Newburgh) *([2], Corollary 3.4.5) Let A be a Banach algebra and $a \in A$. If $\sigma(a)$ is totally disconnected, then $x \mapsto \sigma(x)$ is continuous at a.*

2.5 The socle of a Banach algebra

Definition 2.5.1 (Socle) *([5], Definition 8, p.156) Let A be a Banach algebra. If A has minimal left (right) ideals, then the sum of all minimal left (right) ideals is called the left (right) socle of A. If A has both minimal left and minimal right ideals, and if the left socle coincides with the right socle, then it is called the socle of A, denoted by* $\mathrm{Soc}(A)$. *If A has neither minimal left ideals nor minimal right ideals, then* $\mathrm{Soc}(A) = \{0\}$.

It can easily be shown that $\mathrm{Soc}(A)$ is a two-sided ideal of A. In fact, if A is finite dimensional, then $A = \mathrm{Soc}(A)$.

A Banach algebra A is said to be *semiprime* if $I = \{0\}$ is the only two-sided ideal of A which satisfies $I^2 = \{0\}$.

We proceed with a brief discussion of a class of finite rank elements in a semiprime Banach algebra, introduced by J. Puhl in 1978. If A denotes a semiprime Banach algebra, then $0 \neq a \in A$ is said to be a *spatially rank one element* if there exists a linear functional f_a on A such that

$$axa = f_a(x)a$$

for all $x \in A$.

Definition 2.5.2 (Spatially finite-rank elements) *([23], p.659) Let A be a semiprime Banach algebra. An element $a \in A$ is said to be spatially finite-rank if $a = 0$ or if a is of the form $a = \sum_{i=1}^{n} a_i$, where each a_i is a spatially rank one element and $n \in \mathbb{N}$.*

The set of all spatially finite-rank elements will be denoted by $\mathcal{F}$. In ([23], p.659) J. Puhl showed that, if A is a semiprime Banach algebra, $\mathcal{F} = \mathrm{Soc}(A)$. The following result is also due to J. Puhl.

Proposition 2.5.3 *([23], Corollary 3.5) Let A be a semiprime Banach algebra and $0 \neq a \in A$. Then $a \in \mathcal{F}$ if and only if* $\dim(aAa) < \infty$.

It is easy to show that every semisimple Banach algebra is semiprime ([5], Proposition 5, p.155). Hence, if A is a semisimple Banach algebra, we have from Proposition 2.5.3 that $a \in \mathcal{F} = \mathrm{Soc}(A)$ if and only if $\dim(aAa) < \infty$. This fact implies our next result.

We say that a Banach algebra element a is *algebraic of degree n* if there exists a polynomial p of degree n such that $p(a) = 0$, where n is the smallest integer making this property possible. A set B is then called algebraic if every element of B is algebraic.

Lemma 2.5.4 *Let A be a semisimple Banach algebra. Then* $\mathrm{Soc}(A)$ *is algebraic.*

Proof:
Let $a \in \mathrm{Soc}(A)$. Using the remark following Proposition 2.5.3, suppose that $\dim(aAa) = n$. Then, for each $x \in A$, the elements $\mathbf{1}, axa, (axa)^2, \ldots, (axa)^n$ are linearly independent. Hence, for $x = \mathbf{1}$, we have that there exists scalars $\{\alpha_0, \ldots, \alpha_n\}$, not all zero, such that

$$\alpha_0 \mathbf{1} + \alpha_1 a^2 + \cdots + \alpha_n (a^2)^n = 0.$$

Let $p_a(\lambda) = \alpha_0 + \alpha_1 \lambda^2 + \cdots + \alpha_n \lambda^{2n}$. Then p_a is a non-trivial polynomial with $\deg p_a \leq 2n$ and $p_a(a) = 0$. Since $a \in \mathrm{Soc}(A)$ was arbitrary, the result follows. ∎

The following two results are immediate consequences of Lemma 2.5.4.

Corollary 2.5.5 *Let A be a semisimple Banach algebra. If $a \in \mathrm{Soc}(A) \cap \mathrm{QN}(A)$, then $a \in \mathrm{N}(A)$.*

Proof:
Suppose that $a \in \mathrm{Soc}(A) \cap \mathrm{QN}(A)$. Since $a \in \mathrm{QN}(A)$, we have that $\sigma(a) = \{0\}$ and since $a \in \mathrm{Soc}(A)$, it follows from Lemma 2.5.4 that there exists a polynomial p of degree n (say) such that $p(a) = 0$. By the spectral mapping theorem the constant term in p must be zero. Let m be the smallest power of a in $p(a)$. Then

$$0 = p(a) = a^m(\alpha_m \mathbf{1} + \alpha_{m+1} a + \cdots + \alpha_n a^{n-m}),$$

where $\alpha_m \neq 0$. Again by the spectral mapping theorem, $\sigma(a^k) = \{0\}$ for all $k \geq 1$, and hence $0 \notin \sigma(\alpha_m \mathbf{1} + \alpha_{m+1} a + \cdots + \alpha_n a^{n-m})$, so that $\alpha_m \mathbf{1} + \alpha_{m+1} a + \cdots + \alpha_n a^{n-m} \in A^{-1}$. It then follows that $a^m = 0$; that is, $a \in \mathrm{N}(A)$. ∎

Corollary 2.5.6 *Let A be a semisimple Banach algebra. If $a \in \mathrm{Soc}(A)$, then $\#\sigma(a) < \infty$.*

Proof:
Suppose that $a \in \mathrm{Soc}(A)$. By Lemma 2.5.4, a is algebraic of degree n (say), and hence there exists a polynomial p of degree n such that $p(a) = 0$. By the spectral mapping theorem we have that

$$0 = \sigma(p(a)) = p(\sigma(a)) = \{p(\lambda) : \lambda \in \sigma(a)\}.$$

Hence, every element of $\sigma(a)$ is a root of p. Since $\deg p = n$, we must have that $\#\sigma(a) \leq n$. This completes the proof. ∎

Proposition 2.5.7 *Let A be a semisimple Banach algebra. The spectrum is continuous on $\mathrm{Soc}(A)$.*

Proof:
Let $a \in \mathrm{Soc}(A)$. By Corollary 2.5.6, $\#\sigma(a) < \infty$ and hence $\sigma(a)$ is totally disconnected. Since $a \in \mathrm{Soc}(A)$ was arbitrary, we have from Corollary 2.4.5 that $x \to \sigma(x)$ is continuous on $\mathrm{Soc}(A)$. ∎

2.6 The spectral rank of a semisimple Banach algebra element

In the previous section we briefly discussed spatially finite-rank elements in a semiprime Banach algebra. In 1996, B. Aupetit and H. du T. Mouton defined another type of finite-rank element. We have the following definition.

Definition 2.6.1 (Spectral rank) *([3], p.117) Let A be a semisimple Banach algebra. An element $a \in A$ is said to be a spectrally finite-rank element if there exists a positive integer m such that $\#\sigma'_A(ax) \leq m$, for all $x \in A$. The smallest such m will be called the spectral rank of a, denoted by* $\operatorname{rank}_A(a)$.

By $\mathcal{G}$ we denote the set of spectrally finite-rank elements of a semisimple Banach algebra. Note that Definition 2.6.1 implies that

$$\operatorname{rank}_A(a) = \sup\{\#\sigma'_A(ax) : x \in A\},$$

for $a \in \mathcal{G}$. By the remark following the definition of the spectrum, $\operatorname{rank}_A(a) = \sup\{\#\sigma'_A(xa) : x \in A\}$ is an alternative representation of the spectral rank. If the Banach algebra under discussion is clear from the context, then we will only write $\operatorname{rank}(a)$.

We call an idempotent $p \neq 0$ of a Banach algebra A *minimal* if every nonzero element of pAp is invertible in pAp.

The next result gives a few basic properties of the spectral rank.

Proposition 2.6.2 *([3], p.117) Let A be a semisimple Banach algebra, $a, x \in A$ and $p \in A$ an idempotent. Then*
(a) $\operatorname{rank}(ax) \leq \operatorname{rank}(a)$ *and* $\operatorname{rank}(xa) \leq \operatorname{rank}(a)$,
(b) $\operatorname{rank}(ua) = \operatorname{rank}(a) = \operatorname{rank}(au)$, *if $u \in A^{-1}$*,
(c) $\operatorname{rank}(a) = \operatorname{rank}(\mathbf{1})$, *if $a \in A^{-1}$*,
(d) $\operatorname{rank}(p) = 1$ *if and only if p is minimal.*

Let us remark that the notion of the spectral rank coincides with the notion of the rank of a bounded linear operator T on a Banach space X ([3], p.118).

In the 1993 paper of Mouton and Raubenheimer [18], they indirectly showed the following relationship between spatially finite-rank elements and spectrally finite-rank elements in semisimple Banach algebras. A complete proof can be found in the masters thesis ([20], Theorem 4.4.1) of K. Muzundu.

Theorem 2.6.3 *([18], Theorem 3.1) If A is a semisimple Banach algebra, then $\mathcal{F} = \mathcal{G}$.*

Hence in a semisimple Banach algebra A we have that $\mathcal{F} = \mathcal{G} = \operatorname{Soc}(A)$, which implies that the socle of A can be expressed as $\operatorname{Soc}(A) = \{a \in A : \operatorname{rank}(a) < \infty\}$. Also, since the sets $\mathcal{F}$, $\mathcal{G}$ and $\operatorname{Soc}(A)$ coincide in a semisimple Banach algebra A, we will unambiguously refer to the elements of these sets as elements of the socle of A.

The following result will be required in the proof of Theorem 6.3.5.

Lemma 2.6.4 *Let A be a semisimple Banach algebra and p an idempotent of A. If $b \in pAp \cap \mathrm{Soc}(A)$, then* $\mathrm{rank}_{pAp}(b) = \mathrm{rank}_A(b)$.

Proof:
Suppose that $p = p^2$ and that $b \in pAp \cap \mathrm{Soc}(A)$. Let $x \in A$ be such that $b = pxp$. By using Lemma 2.2.9 and the remark following the definition of the spectrum we have that

$$\begin{aligned}
\mathrm{rank}_A(b) &= \sup\{\#\sigma'_A(by) : y \in A\} \\
&= \sup\{\#\sigma'_A(pxpy) : y \in A\} \\
&= \sup\{\#\sigma'_A(p(pxpy)) : y \in A\} \\
&= \sup\{\#\sigma'_A((pxpy)p) : y \in A\} \\
&= \sup\{\#\sigma'_{pAp}(pxp^2yp) : y \in A\} \\
&= \sup\{\#\sigma'_{pAp}(bpyp) : y \in A\} = \mathrm{rank}_{pAp}(b)
\end{aligned}$$

■

Lemma 2.6.4 implies that if $b \in pAp \cap \mathrm{Soc}(A)$, where p is an idempotent of a semisimple Banach algebra A, then we may refer unambiguously to $\mathrm{rank}(b)$.

Definition 2.6.5 *([3], p.118) Let A be a semisimple Banach algebra and $a \in \mathrm{Soc}(A)$. The set $E(a)$ is defined by*

$$E(a) = \{x \in A : \mathrm{rank}(a) = \#\sigma'(xa)\}.$$

By the definition of the spectral rank, $E(a) \neq \emptyset$. The set $E(a)$ gives rise to the following definition.

Definition 2.6.6 (Maximal finite-rank) *([3], p.118) Let A be a semisimple Banach algebra. An element $a \in \mathrm{Soc}(A)$ is called a maximal finite-rank element if* $\mathrm{rank}(a) = \#\sigma'(a)$.

The following result will be useful in the proof of Theorem 6.3.5.

Theorem 2.6.7 (Density of maximal finite-rank elements) *([3], Theorem 2.2) Let A be a semisimple Banach algebra and $a \in \mathrm{Soc}(A)$. Then $E(a)$ is a dense open subset of A.*

Theorem 2.6.8 (Diagonalization theorem) *([3], Theorem 2.8) Let A be a semisimple Banach algebra and $0 \neq a \in \mathrm{Soc}(A)$. If a is a maximal finite-rank element and $\sigma'(a) = \{\lambda_1, \ldots, \lambda_n\}$, then $a = \lambda_1 p_1 + \cdots + \lambda_n p_n$, where $p_1, \ldots, p_n$ are orthogonal minimal idempotents.*

In the following two results we present a few basic properties of the rank of a matrix that will be required in the proof of Theorem 6.3.5.

Proposition 2.6.9 *([14], Theorems 3.17 and 3.18) Let $A \in M_n(\mathbb{C})$. Then*
(a) $\mathrm{rank}(A) \leq n$,
(b) $\mathrm{rank}(A) = n$ *if and only if* $A \in M_n(\mathbb{C})^{-1}$.

Lemma 2.6.10 *Let $A, B \in M_n(\mathbb{C})$. If $A \in M_n(\mathbb{C})^{-1}$, AB is maximal finite-rank and* $\mathrm{rank}(A) = \#\sigma'(AB)$, *then $B \in M_n(\mathbb{C})^{-1}$.*

Proof:
Suppose that $A \in M_n(\mathbb{C})^{-1}$, AB is maximal finite-rank and that $\mathrm{rank}(A) = \#\sigma'(AB)$. Then, by hypothesis and Propositions 2.6.2(b) and 2.6.9(b), we have that

$$\mathrm{rank}(B) = \mathrm{rank}(AB) = \#\sigma'(AB) = \mathrm{rank}(A) = n.$$

Hence, using Proposition 2.6.9(b) again, we have that $B \in M_n(\mathbb{C})^{-1}$. ■

2.7 Continuity of inversion in a Banach algebra

Theorem 2.7.1 *([2], Theorem 3.2.3) Let A be a Banach algebra and $a \in A^{-1}$. If $||x - a|| < \frac{1}{||a^{-1}||}$, then $x \in A^{-1}$. Moreover, the mapping $x \mapsto x^{-1}$ is continuous from A^{-1} onto A^{-1}.*

The map in Theorem 2.7.1 will be called *inversion*. This theorem implies that $\mathcal{B}(a, ||a^{-1}||^{-1}) \subseteq A^{-1}$ for all $a \in A^{-1}$, indicating that the set A^{-1} is open.

The following two lemmas are well-known continuity results for invertibility.

Lemma 2.7.2 *([25], (A), p.207) Let A be a Banach algebra, (a_n) a convergent sequence in A with limit a, and suppose all a_n are invertible. Then the following statements are equivalent:*
(a) $\sup\{||a_n^{-1}|| : n \in \mathbb{N}\} < \infty$.
(b) a is invertible and the sequence (a_n^{-1}) is convergent with limit a^{-1}.

Proof:
Suppose that $a_n \to a$ as $n \to \infty$, where all a_n are invertible in A. For the non-trivial implication, suppose that (a) holds. Let $C = \sup\{||a_n^{-1}|| : n \in \mathbb{N}\}$. Then $C > 0$ and for all $n \in \mathbb{N}$ we have that $||a_n^{-1}|| \leq C$. Since $a_n \to a$ as $n \to \infty$, we can find a positive integer N such that $||a_n - a|| < C^{-1}$, for all $n \geq N$. Hence, for all $n \geq N$,

$$||a_n^{-1}(a - a_n)|| \leq ||a_n^{-1}||||a - a_n|| < CC^{-1} = 1.$$

By Theorem 2.1.9, $\mathbf{1} + a_n^{-1}(a - a_n) \in A^{-1}$, so that $a = a_n[\mathbf{1} + a_n^{-1}(a - a_n)] \in A^{-1}$. Hence $a_n \to a$ in A^{-1} as $n \to \infty$, so that $a_n^{-1} \to a^{-1}$ as $n \to \infty$ by Theorem 2.7.1. ■

Lemma 2.7.3 *([25], (B), p.208) Let A be a Banach algebra and (a_n) a convergent sequence in A with limit a. If $a \in A^{-1}$, then $a_n \in A^{-1}$ for all sufficiently large n, and $a_n^{-1} \to a^{-1}$ as $n \to \infty$.*

Proof:
Suppose that $a_n \to a \in A^{-1}$ as $n \to \infty$. Since A^{-1} is open, we can find an $r > 0$ such that $\mathcal{B}(a, r) \subseteq A^{-1}$. Since $a_n \to a$ as $n \to \infty$, there exists an $N \in \mathbb{N}$ such that, for all $n \geq N$, the inequality $||a_n - a|| < r$ holds, that is, $a_n \in \mathcal{B}(a, r)$. Hence, for all $n \geq N$, (a_n) is a sequence of invertible elements which converge to a in A^{-1}. By Theorem 2.7.1, $a_n^{-1} \to a^{-1}$ as $n \to \infty$. ∎

2.8 Bounded linear operators on Banach spaces

Recall that $\mathfrak{L}(X)$ denotes the Banach algebra of all bounded linear operators on a complex Banach space X. Let $T \in \mathfrak{L}(X)$. The set of all $x \in X$ such that $Tx = 0$ is called the *null space* of T. The *range* of T is the set of all Tx with $x \in X$. We use the notations $\mathrm{Null}(T)$ and $\mathrm{R}(T)$ to denote the null space and the range of T, respectively. It is well-known that $\mathrm{Null}(T)$ is closed.

Definition 2.8.1 (Ascent, descent) *([19], p.178) Let $T \in \mathfrak{L}(X)$. The ascent (descent) of T is the smallest non-negative integer k such that $\mathrm{Null}(T^k) = \mathrm{Null}(T^{k+1})$ ($\mathrm{R}(T^k) = \mathrm{R}(T^{k+1})$). We write $\mathrm{asc}(T)$ and $\mathrm{des}(T)$ to denote the ascent and descent of T, respectively.*

We need the following result in Chapter 7.

Theorem 2.8.2 *([19], Theorem 4, Corollary 5, p.179) Let $T \in \mathfrak{L}(X)$. If $\mathrm{asc}(T)$ and $\mathrm{des}(T)$ are finite, then $\mathrm{asc}(T) = \mathrm{des}(T) = k$ (say) and hence $\mathrm{R}(T^k)$ is closed and $X = \mathrm{Null}(T^k) \oplus \mathrm{R}(T^k)$. Also, $\mathrm{Null}(T^k)$ and $\mathrm{R}(T^k)$ are invariant subspaces of T, that is, $T(\mathrm{Null}(T^k)) \subseteq \mathrm{Null}(T^k)$ and $T(\mathrm{R}(T^k)) \subseteq \mathrm{R}(T^k)$. Moreover, $T^k_{|\mathrm{Null}(T^k)} = 0$ and $T_{|\mathrm{R}(T^k)} \in \mathfrak{L}(\mathrm{R}(T^k))^{-1}$.*

If X is a normed space, then the *dual* of X, which is the set of all bounded linear functionals on X, will be denoted by X^*.

The *adjoint* of $T \in \mathfrak{L}(X)$, denoted by T^*, is the unique operator $T^* : X^* \to X^*$ satisfying

$$(T^*g)(x) = g(Tx),$$

for all $x \in X$ and $g \in X^*$. It is well-known that $||T|| = ||T^*||$. The following two useful properties can be easily shown:

$$\mathrm{Null}(T^*) = \mathrm{R}(T)^\perp = \{f \in X^* : f(y) = 0 \text{ for all } y \in \mathrm{R}(T)\}$$

and

$$\mathrm{R}(T^*) = \mathrm{Null}(T)^\perp = \{f \in X^* : f(y) = 0 \text{ for all } y \in \mathrm{Null}(T)\}.$$

Definition 2.8.3 (Minimum modulus) *([19], Definition 3, p.86) Let X and Y be Banach spaces and $T : X \to Y$ a bounded linear operator. We define the minimum modulus $j(T)$ of T by*

$$j(T) := \inf\{||Tx|| : x \in X, ||x|| = 1\}.$$

It can be easily shown that

$$j(T) = \inf\left\{\frac{||Tx||}{||x||} : x \in X, x \neq 0\right\}$$

is an alternative formula for the minimum modulus. It is also clear from the definition of the minimum modulus that $j(T) \leq ||T||$.

Finally, the following result will be needed in Chapter 6.

Example 2.8.4 *([19], p.93) Let H be a Hilbert space with an orthonormal basis $\{e_{i\geq 0}\}$. If $w_i \geq 0$ and $T : H \to H$ is defined by $Te_i = w_i e_{i+1}$, then $\sigma_{\mathfrak{L}(H)}(T) = \overline{\mathcal{B}}(0, r(T)) := \{\lambda \in \mathbb{C} : |\lambda| \leq r(T)\}$.*

The operator in Example 2.8.4 is called the *weighted unilateral shift* with weights $w_i \geq 0$.

2.9 Reduced minimum modulus

Definition 2.9.1 (Reduced minimum modulus) *([19], p.97) Let $T \in \mathfrak{L}(X)$. We define the reduced minimum modulus of T, denoted by $\gamma(T)$, by*

$$\gamma(T) = \begin{cases} \inf\{||Tu|| : u \in X, \mathrm{D}(u, \mathrm{Null}(T)) = 1\} & \text{if } T \neq 0 \\ \infty & \text{if } T = 0. \end{cases}$$

Lemma 2.9.2 *([19], Theorem 3, p.97) If $T \in \mathfrak{L}(X)$, then $\gamma(T) = \gamma(T^*)$.*

If $T \in \mathfrak{L}(X)$, we denote by $\tilde{T}$ the operator $\tilde{T} : X/\mathrm{Null}(T) \to \overline{\mathrm{R}(T)}$, defined by

$$\tilde{T}(x + \mathrm{Null}(T)) = Tx.$$

It is clear that $\mathrm{R}(\tilde{T}) = \mathrm{R}(T)$; hence the range of $\tilde{T}$ is dense in $\overline{\mathrm{R}(T)}$. An important property of $\tilde{T}$ is that it is one-to-one. Therefore, if T has closed range, then $\tilde{T}$ is invertible. Also, if T is invertible, then $\tilde{T} = T$.

It is easy to see, from the definition of the reduced minimum modulus, that $\gamma(T) = j(T)$ whenever T is one-to-one. In general we have the following relationship between the minimum modulus and the reduced minimum modulus.

Lemma 2.9.3 *([19], Lemma 1, p.97) If $T \in \mathfrak{L}(X)$, then $\gamma(T) = j(\tilde{T})$.*

Lemma 2.9.4 *([19], Theorem 7, p.87) If $T \in \mathfrak{L}(X)$ has closed range, then $\gamma(T) = ||\tilde{T}^{-1}||^{-1}$. Hence, if $T \in \mathfrak{L}(X)^{-1}$, then $\gamma(T) = ||T^{-1}||^{-1}$.*

Proof:
Suppose that $T \in \mathfrak{L}(X)$ has closed range. Using Lemma 2.9.3 and the fact that $\tilde{T}$ is invertible, we have that

$$\begin{aligned}
\gamma(T) = j(\tilde{T}) &= \inf\left\{\frac{||\tilde{T}(x + \mathrm{Null}(T))||}{||x + \mathrm{Null}(T)||_{X/\mathrm{Null}(T)}} : x \notin \mathrm{Null}(T)\right\} \\
&= \left(\sup\left\{\frac{||x + \mathrm{Null}(T)||_{X/\mathrm{Null}(T)}}{||\tilde{T}(x + \mathrm{Null}(T))||} : x \notin \mathrm{Null}(T)\right\}\right)^{-1} \\
&= \left(\sup\left\{\frac{||\tilde{T}^{-1}(Tx)||_{X/\mathrm{Null}(T)}}{||Tx||} : x \notin \mathrm{Null}(T)\right\}\right)^{-1} \\
&= ||\tilde{T}^{-1}||^{-1}.
\end{aligned}$$

■

The following result gives a relation between the reduced minimum modulus and the closedness of the range of a bounded linear operator.

Theorem 2.9.5 *([19], Theorem 2, p.97) Let $T \in \mathfrak{L}(X)$. Then $\mathrm{R}(T)$ is closed if and only if $\gamma(T) > 0$.*

2.10 The gap between closed subspaces

In this section we introduce the notion of the gap between two closed subspaces of a Banach space.

Definition 2.10.1 (Gap) *([19], Definition 5, p.98) Let X be a complex Banach space and let $M, N \subseteq X$ be closed subspaces. We define the gap between M and N by*

$$\mathrm{gap}(M, N) = \max\{\delta(M, N), \delta(N, M)\},$$

where

$$\delta(M, N) = \begin{cases} \sup\{\mathrm{D}(u, N) : u \in M, ||u|| = 1\} & \text{if } M \neq \{0\} \\ 0 & \text{if } M = \{0\}. \end{cases}$$

It is clear from the definition of the gap that $\mathrm{gap}(M, N) = \mathrm{gap}(N, M)$. We also have, from the definition of δ, that $\delta(M, N) = 1$ if $N = \{0\}$ and $M \neq \{0\}$, and $\delta(M, N) = 0$ whenever $M \subseteq N$.

The gap can be seen as a function which measures the "distance" between two subspaces. Note that the gap is not a proper distance function, since it does not in general satisfy the triangle inequality required to be a distance function ([19], Lemma 6, p.98).

The following result will be required in Chapter 7.

Theorem 2.10.2 *([19], Theorem 8, p.99) Let X be a complex Banach space and let $M, N \subseteq X$ be closed subspaces. Then* $\operatorname{gap}(M, N) = \operatorname{gap}(M^{\perp}, N^{\perp})$.

We will continue our discussion on the gap in Chapter 7.

2.11 Connections between the gap and the reduced minimum modulus

In this section we formulate and prove useful relations between the reduced minimum modulus and the gap function (Lemma 2.11.3). In order to do so, we need the following simple auxiliary result. We will, as a result of Theorem 2.9.5, only consider bounded linear operators with closed range.

Lemma 2.11.1 *Let $T \in \mathfrak{L}(X)$ be a closed range operator and s a positive number strictly less that $\gamma(T)$. The following statements are true:*
(a) $||x + \mathrm{Null}(T)||_{X/\mathrm{Null}(T)} < \frac{1}{s}$ for all $x \in X$ with $||Tx|| = 1$.
(b) If $x \in X$ satisfies $||Tx|| = 1$, then there exists an element $x_0 \in X$ such that $Tx_0 = Tx$ and $||x_0|| \leq \frac{1}{s}$.
(c) If $x \in X$, then there exists an element $x_1 \in X$ such that $Tx_1 = Tx$ and $||x_1|| \leq \frac{1}{s}||Tx||$.

Proof:
Suppose that $\mathrm{R}(T)$ is closed and $s > 0$ is such that $s < \gamma(T)$.
(a) By Lemma 2.9.4 we have that $||\tilde{T}^{-1}|| = \gamma(T)^{-1} < s^{-1}$, and hence $||\tilde{T}^{-1}y|| < s^{-1}$ for all $y \in \mathrm{R}(T)$ with $||y|| = 1$. Set $y = Tx$. It then follows that for all $x \in X$ with $||Tx|| = 1$ we have that $||x + \mathrm{Null}(T)|| < \frac{1}{s}$.

(b) Let $x \in X$ be such that $||Tx|| = 1$. From (a) it follows that $||x + \mathrm{Null}(T)||_{X/\mathrm{Null}(T)} < \frac{1}{s}$. Let $\epsilon > 0$ be such that $||x + \mathrm{Null}(T)||_{X/\mathrm{Null}(T)} + \epsilon = \frac{1}{s}$. By the definition of the greatest lower bound there exists $y \in \mathrm{Null}(T)$ such that $||x + y|| < ||x + \mathrm{Null}(T)||_{X/\mathrm{Null}(T)} + \epsilon = \frac{1}{s}$. Let $x_0 = x + y$. Then $Tx_0 = T(x + y) = Tx$ and $||x_0|| \leq \frac{1}{s}$.

(c) If $x \in \mathrm{Null}(T)$, then $x_1 = 0$ works. Now suppose that $x \notin \mathrm{Null}(T)$. Then $\left|\left|T\left(\frac{x}{||Tx||}\right)\right|\right| = 1$. By (b) there exists an $x_0 \in X$ such that $Tx_0 = T\left(\frac{x}{||Tx||}\right)$ and $||x_0|| \leq \frac{1}{s}$. Set $x_1 = ||Tx||x_0$. It then follows that $Tx_1 = Tx$ and $||x_1|| = ||Tx||||x_0|| \leq \frac{1}{s}||Tx||$. ■

The following result will be required in the proof of Lemma 2.11.3. We will, however, not prove this result, since a complete proof can be found in the book of Müller (see [19]).

Lemma 2.11.2 *([19], Lemma 13, p.101) Let $A, B \in \mathfrak{L}(X)$ be closed range operators. If $\delta(\mathrm{Null}(A), \mathrm{Null}(B)) < \frac{1}{2}$, then*

$$\gamma(A) \leq \frac{||A - B|| + \gamma(B)}{1 - 2\delta(\mathrm{Null}(A), \mathrm{Null}(B))}.$$

Let us remark that our next result is Lemma 3.4 in [13]. It was originally proved in [16] but, since this reference may not be readily available, we will supply a proof. For assertions (1) and (2) in Lemma 2.11.3 we follow along the same lines as that of ([19], Lemma 12, p.101) and rely on Lemma 2.11.1. We use Lemma 2.11.2 in the proof of statement (3). Lemma 2.11.3 will then in turn be used in the proof of Lemma 2.11.4. Let us mention that, for $T \in \mathfrak{L}(X)$, $\mathrm{R}(T)$ is closed if and only if $\mathrm{R}(T^*)$ is closed ([19], Theorem 16, p.396).

Lemma 2.11.3 *([16], p.268-269) Let $A, B \in \mathfrak{L}(X)$ be closed range operators. Then*
(1) $\mathrm{gap}(\mathrm{Null}(A), \mathrm{Null}(B)) \leq \max\left\{\frac{1}{\gamma(A)}, \frac{1}{\gamma(B)}\right\} ||A-B||$,

(2) $\mathrm{gap}(\mathrm{R}(A), \mathrm{R}(B)) \leq \max\left\{\frac{1}{\gamma(A)}, \frac{1}{\gamma(B)}\right\} ||A-B||$,

(3) $|\gamma(A)-\gamma(B)| \leq \frac{3||A-B||}{1-2\mathrm{gap}(\mathrm{Null}(A),\mathrm{Null}(B))}$, *whenever* $\mathrm{gap}(\mathrm{Null}(A), \mathrm{Null}(B)) < \frac{1}{2}$,

(4) $|\gamma(A)-\gamma(B)| \leq \frac{3||A-B||}{1-2\mathrm{gap}(\mathrm{R}(A),\mathrm{R}(B))}$, *whenever* $\mathrm{gap}(\mathrm{R}(A), \mathrm{R}(B)) < \frac{1}{2}$.

Proof:
Suppose that $\mathrm{R}(A)$ and $\mathrm{R}(B)$ are closed in X.
(1) Since $\mathrm{R}(A)$ is closed, $\gamma(A) > 0$ by Theorem 2.9.5. Let s be a positive number such that $s < \gamma(A)$. If $\mathrm{Null}(B) = \{0\}$, then

$$\delta(\mathrm{Null}(B), \mathrm{Null}(A)) = 0 \leq \gamma(A)^{-1}||A-B||.$$

Now suppose that $\mathrm{Null}(B) \neq \{0\}$ and let $x \in \mathrm{Null}(B)$ with $||x|| = 1$. Then $||Ax|| = ||(A-B)x|| \leq ||A-B||$. By Lemma 2.11.1(c) we have that there exists an $x_1 \in X$ such that $Ax_1 = Ax$ and $||x_1|| \leq s^{-1}||Ax|| \leq s^{-1}||A-B||$. Since $x - x_1 \in \mathrm{Null}(A)$, we have that

$$\mathrm{D}(x, \mathrm{Null}(A)) \leq ||x-(x-x_1)|| = ||x_1|| \leq s^{-1}||A-B||,$$

so that $\delta(\mathrm{Null}(B), \mathrm{Null}(A)) \leq s^{-1}||A-B||$. Since s in $0 < s < \gamma(A)$ was arbitrary, it follows that

$$\delta(\mathrm{Null}(B), \mathrm{Null}(A)) \leq \gamma(A)^{-1}||A-B||.$$

Analogously, using the fact that $\mathrm{R}(B)$ is closed, we have that

$$\delta(\mathrm{Null}(A), \mathrm{Null}(B)) \leq \gamma(B)^{-1}||A-B||.$$

Now,

$$\delta(\mathrm{Null}(B), \mathrm{Null}(A)) \leq \gamma(A)^{-1}||A-B|| \leq \max\left\{\frac{1}{\gamma(A)}, \frac{1}{\gamma(B)}\right\} ||A-B||$$

and

$$\delta(\mathrm{Null}(A), \mathrm{Null}(B)) \leq \gamma(B)^{-1}||A-B|| \leq \max\left\{\frac{1}{\gamma(A)}, \frac{1}{\gamma(B)}\right\}||A-B||,$$

so that $\mathrm{gap}(\mathrm{Null}(A), \mathrm{Null}(B)) \leq \max\left\{\frac{1}{\gamma(A)}, \frac{1}{\gamma(B)}\right\}||A-B||$.

(2) Since $\mathrm{R}(A)$ is closed, $\gamma(A) > 0$ by Theorem 2.9.5. Let s be a positive number such that $s < \gamma(A)$. If $\mathrm{R}(A) = \{0\}$, then

$$\delta(\mathrm{R}(A), \mathrm{R}(B)) = 0 \leq \gamma(A)^{-1}||A-B||.$$

Now suppose that $\mathrm{R}(A) \neq \{0\}$ and let $y \in \mathrm{R}(A)$ with $||y|| = 1$. Then there exists an element $x \in X$ such that $Ax = y$. We also have from Lemma 2.11.1(c) that there exists an element $x_1 \in X$ such that $Ax_1 = Ax$ and $||x_1|| \leq s^{-1}||Ax|| = s^{-1}$. Hence

$$\mathrm{D}(y, \mathrm{R}(B)) \leq ||y - Bx_1|| = ||(A-B)x_1|| \leq ||A-B||||x_1|| \leq s^{-1}||A-B||,$$

so that $\delta(\mathrm{R}(A), \mathrm{R}(B)) \leq s^{-1}||A-B||$. Since s in $0 < s < \gamma(A)$ was arbitrary, it follows that

$$\delta(\mathrm{R}(A), \mathrm{R}(B)) \leq \gamma(A)^{-1}||A-B||.$$

A similar argument, using the fact that $\mathrm{R}(B)$ is closed, shows that

$$\delta(\mathrm{R}(B), \mathrm{R}(A)) \leq \gamma(B)^{-1}||A-B||.$$

Hence

$$\mathrm{gap}(\mathrm{R}(A), \mathrm{R}(B)) \leq \max\left\{\frac{1}{\gamma(A)}, \frac{1}{\gamma(B)}\right\}||A-B||.$$

(3) Suppose that $\mathrm{gap}(\mathrm{Null}(A), \mathrm{Null}(B)) < \frac{1}{2}$. It suffices to show that

$$\gamma(A) \leq \frac{3||A-B||}{1 - 2\mathrm{gap}(\mathrm{Null}(A), \mathrm{Null}(B))} + \gamma(B). \tag{2.11.1}$$

If $\gamma(A) < \gamma(B)$, then $\gamma(A) - \gamma(B) < 0 \leq \frac{3||A-B||}{1-2\mathrm{gap}(\mathrm{Null}(A),\mathrm{Null}(B))}$, and hence (2.11.1) holds.

If $\gamma(A) \geq \gamma(B)$, then $\max\left\{\frac{1}{\gamma(A)}, \frac{1}{\gamma(B)}\right\} = \frac{1}{\gamma(B)}$. By (1) we have that

$$\gamma(B)\mathrm{gap}(\mathrm{Null}(A), \mathrm{Null}(B)) \leq ||A-B||$$

and hence, by also using Lemma 2.11.2, it follows that

$$\begin{aligned}
\gamma(A)-\gamma(B) &\leq \frac{||A-B||+\gamma(B)}{1-2\delta(\mathrm{Null}(A),\mathrm{Null}(B))}-\gamma(B)\\
&= \frac{||A-B||+\gamma(B)-\gamma(B)[1-2\delta(\mathrm{Null}(A),\mathrm{Null}(B))]}{1-2\delta(\mathrm{Null}(A),\mathrm{Null}(B))}\\
&\leq \frac{||A-B||+2\gamma(B)\mathrm{gap}(\mathrm{Null}(A),\mathrm{Null}(B))}{1-2\delta(\mathrm{Null}(A),\mathrm{Null}(B))}\\
&\leq \frac{||A-B||+2||A-B||}{1-2\delta(\mathrm{Null}(A),\mathrm{Null}(B))}\\
&\leq \frac{3||A-B||}{1-2\mathrm{gap}(\mathrm{Null}(A),\mathrm{Null}(B))}.
\end{aligned}$$

Hence (2.11.1) holds and the result then follows.

(4) Suppose that $\mathrm{gap}(\mathrm{R}(A),\mathrm{R}(B))<\frac{1}{2}$. It then follows from Theorem 2.10.2 that

$$\begin{aligned}
\mathrm{gap}(\mathrm{Null}(A^*),\mathrm{Null}(B^*)) &= \mathrm{gap}(\mathrm{R}(A)^\perp,\mathrm{R}(B)^\perp)\\
&= \mathrm{gap}(\mathrm{R}(A),\mathrm{R}(B))<\frac{1}{2}.
\end{aligned}$$

Since A^* and B^* are closed range operators, it follows from (3) together with Lemma 2.9.2 that

$$\begin{aligned}
|\gamma(A)-\gamma(B)|=|\gamma(A^*)-\gamma(B^*)| &\leq \frac{3||A^*-B^*||}{1-2\mathrm{gap}(\mathrm{Null}(A^*),\mathrm{Null}(B^*))}\\
&= \frac{3||A-B||}{1-2\mathrm{gap}(\mathrm{R}(A),\mathrm{R}(B))}.
\end{aligned}$$

This completes the proof. ■

We immediately have the following lemma. Take note that Lemma 2.11.4 is Lemma 3.5 in [13]. Just like the previous lemma, this result was originally proved by Markus in [16]. Lemma 2.11.4 will be useful in Chapter 7.

Lemma 2.11.4 *([16], Theorem 2, Remark 1) Let $T_n, T\in\mathfrak{L}(X)$ be closed range operators such that $T_n\to T$ as $n\to\infty$. The following statements are equivalent:*
(a) $\inf\{\gamma(T_n):n\in\mathbb{N}\}>0$
(b) $\gamma(T_n)\to\gamma(T)$ *as* $n\to\infty$
(c) $\mathrm{gap}(\mathrm{R}(T_n),\mathrm{R}(T))\to 0$ *as* $n\to\infty$
(d) $\mathrm{gap}(\mathrm{Null}(T_n),\mathrm{Null}(T))\to 0$ *as* $n\to\infty$

Proof:
Suppose that T_n and T are closed range operators in $\mathfrak{L}(X)$ such that $T_n\to T$ as $n\to\infty$.

(a)⇒(c): Suppose that $\inf\{\gamma(T_n) : n \in \mathbb{N}\} > 0$. By Lemma 2.11.3(2) we have that $\text{gap}(\mathrm{R}(T_n), \mathrm{R}(T)) \leq \max\left\{\frac{1}{\gamma(T_n)}, \frac{1}{\gamma(T)}\right\} ||T_n - T||$. The result will follow if we can show that $\max\left\{\frac{1}{\gamma(T_n)}, \frac{1}{\gamma(T)}\right\}$ is bounded. Let $N = \inf\{\gamma(T_n) : n \in \mathbb{N}\} > 0$. Then $\frac{1}{\gamma(T_n)} \leq \frac{1}{N}$, for all $n \in \mathbb{N}$, so that $\max\left\{\frac{1}{\gamma(T_n)}, \frac{1}{\gamma(T)}\right\} \leq \max\left\{\frac{1}{N}, \frac{1}{\gamma(T)}\right\}$. By our assumption $T_n \to T$ as $n \to \infty$; hence $\text{gap}(\mathrm{R}(T_n), \mathrm{R}(T)) \to 0$ as $n \to \infty$.

(a)⇒(d): The proof is similar to the above implication using Lemma 2.11.3(1).

(d)⇒(b): Suppose that $\text{gap}(\text{Null}(T_n), \text{Null}(T)) \to 0$ as $n \to \infty$. Then there exists $n_0 \in \mathbb{N}$ such that $\text{gap}(\text{Null}(T_n), \text{Null}(T)) < \frac{1}{4}$, for all $n \geq n_0$. Let $0 < \epsilon < 1$. Since $T_n \to T$ as $n \to \infty$, we can find an $n_1 \in \mathbb{N}$ such that $||T_n - T|| < \frac{1}{6}\epsilon$, for all $n \geq n_1$. Let $N = \max\{n_0, n_1\}$. Now, by Lemma 2.11.3 (3) we have that $|\gamma(T_n) - \gamma(T)| \leq \frac{3||T_n - T||}{1 - 2\text{gap}(\text{Null}(T_n), \text{Null}(T))} < \epsilon$, for all $n \geq N$. Hence $\gamma(T_n) \to \gamma(T)$ as $n \to \infty$.

(c)⇒(b): : A similar argument as in the above implication can be used, together with Lemma 2.11.3 (4).

(b)⇒(a): Suppose that $\gamma(T_n) \to \gamma(T)$ as $n \to \infty$. Since $\mathrm{R}(T_n)$ is closed for all n, we have that $\gamma(T_n) > 0$ for all n, and hence $\inf\{\gamma(T_n) : n \in \mathbb{N}\} \geq 0$. If $\inf\{\gamma(T_n) : n \in \mathbb{N}\} = 0$, then by the definition of the greatest lower bound, for every $k \in \mathbb{N}$, there exists a T_{n_k} such that $\gamma(T_{n_k}) < \inf\{\gamma(T_n) : n \in \mathbb{N}\} + \frac{1}{k} = \frac{1}{k}$. Hence $\gamma(T_{n_k}) \to 0$ as $k \to \infty$. But this contradicts the fact that $\gamma(T) > 0$; hence $\inf\{\gamma(T_n) : n \in \mathbb{N}\} > 0$. ■

2.12 The Drazin inverse in $M_n(\mathbb{C})$ and $\mathfrak{L}(X)$

Recall that $M_n(\mathbb{C})$ denotes the algebra of all $n \times n$ matrices with complex entries and $\mathfrak{L}(X)$ denotes the algebra of all bounded linear operators on a complex Banach space X. The Drazin inverses of the elements of $M_n(\mathbb{C})$ and $\mathfrak{L}(X)$ have been intensively studied by authors such as A. Ben-Israel and T. N. E. Greville ([4]) and King ([11]), respectively. In this section we present some of the well-known results obtained for the Drazin inverse of a square matrix and the Drazin inverse of a bounded linear operator that will be used, and in some instances generalized to arbitrary Banach algebra elements, in subsequent chapters.

Definition 2.12.1 (Drazin inverse of a square matrix) *([4], p.152) An element $B \in M_n(\mathbb{C})$ is said to have a Drazin inverse if there exists an element $X \in M_n(\mathbb{C})$ and $k \in \mathbb{N}$ such that:*

- $BX = XB$
- $XBX = X$

- $B^k X B = B^k$

The matrix X in the definition above is unique ([4], Theorem 7, p.164) and will be called the *Drazin inverse* of B. We use the notation B^d to denote the Drazin inverse of B. Let us also mention that every square matrix is Drazin invertible.

If, however, $k = 1$ in Definition 2.12.1, then X is called the *group inverse* of A. The terminology was given since the positive powers of a given matrix and its group inverse constitute an Abelian group. We have the following well-known results for matrices in $M_n(\mathbb{C})$.

Theorem 2.12.2 *([4], Theorem 2, p.156) An element $B \in M_n(\mathbb{C})$ has a group inverse if and only if* $\operatorname{rank}(B) = \operatorname{rank}(B^2)$.

Theorem 2.12.3 (Core-nilpotent decomposition of a square matrix) *([4], Theorem 11, p.169) Every $B \in M_n(\mathbb{C})$ has a unique decomposition of the form $B = C + N$, where C has a group inverse, N is nilpotent and $NC = CN = 0$. Moreover, C is the group inverse of B^d.*

We call C in Theorem 2.12.3 the *core matrix* of B and denote it by $B^{(c)}$.

In [8] Campbell and Meyer proved the following result for the continuity of the Drazin inverse of a square matrix.

Theorem 2.12.4 (Campbell and Meyer) *([8], Theorem 10.7.1) Let $B_n, B \in M_n(\mathbb{C})$ be such that $B_n \to B$ as $n \to \infty$. Then the following statements are equivalent:*
(a) $B_n^d \to B^d$ as $n \to \infty$.
(b) There exists $n_0 \in \mathbb{N}$ such that $\operatorname{rank}\left(B_n^{(c)}\right) = \operatorname{rank}\left(B^{(c)}\right)$ *for all $n \geq n_0$.*

Following is the definition of a Drazin invertible bounded linear operator.

Definition 2.12.5 (Drazin inverse of a bounded linear operator) *([11], p.383) Let $T \in \mathfrak{L}(X)$. An element $T \in \mathfrak{L}(X)$ is said to have a Drazin inverse if there exists $S \in \mathfrak{L}(X)$ and $k \in \mathbb{N}$ such that:*

- $TS = ST$
- $STS = S$
- $T^k ST = T^k$

The following result will be required in Chapter 7.

Theorem 2.12.6 *([11], Theorem 4) Let $T \in \mathfrak{L}(X)$. Then T has a Drazin inverse if and only if it has finite ascent and descent. In such a case, if k is the smallest integer for which T has a Drazin inverse, then* $\operatorname{asc}(T) = \operatorname{des}(T) = k$.

Chapter 3

Group inverses

The objective of this chapter is to introduce the concept of a group inverse in an algebra and prove some of the properties of these elements. Most of the work done in this chapter comes from the paper (see [25]) by Roch and Silbermann. This chapter is divided into two sections. In Section 3.1 we investigate properties like the existence and uniqueness of the group inverse in an algebra. The aim of Section 3.2 is to discuss the spectrum of a group invertible element. The main results in this chapter are Proposition 3.1.4, Proposition 3.1.6 and Proposition 3.2.4.

3.1 Introducing the group inverse in algebras

We start by first introducing the concept of a regular element, which is a generalization of the concept of an inverse. Our reason for introducing this notion is to showcase the following advantage group invertibility has over regular elements: the uniqueness of the group inverse provided it exists (Proposition 3.1.4).

Definition 3.1.1 (Regular) *([10], Definition 7.3.1) Let A be an algebra. An element $a \in A$ is said to be regular if there exists an element $b \in A$ such that*

$$a - aba = 0.$$

We will call b a *regular inverse* of a. It is clear from Definition 3.1.1 that every invertible element is regular. Hence, the concept of a regular element generalizes the concept of an inverse. We continue by introducing yet another generalization of invertibility.

Definition 3.1.2 (Group inverse) *([25], p.197) Let A be an algebra. An element $a \in A$ is said to be group invertible if there exists an element $b \in A$ such that:*

- $ab = ba$

- $bab = b$
- $a - aba = 0$

We call b in Definition 3.1.2 a *group inverse* of a. We will see later (Proposition 3.1.4) that group inverses are actually unique. It is obvious that if b is a group inverse of a, then a is a group inverse of b, that is, group inverses are symmetric. It is also clear from the definition of a group inverse that every idempotent is group invertible with itself as a group inverse and that group invertible elements are regular. Let A^g denote the set of all group invertible elements. It is easy to verify that the equations in Definition 3.1.2 hold when a is invertible (choose $b = a^{-1}$). Hence $A^{-1} \subsetneq A^g$. This inclusion is strict since $0 \in A^g$, but $0 \notin A^{-1}$.

We state and prove the following simple useful lemma for group inverses.

Lemma 3.1.3 *Let A be an algebra and $a, b \in A$.*
(1) If a is group invertible with a group inverse b, then a^k is group invertible with a group inverse b^k.
(2) If $bab = b$, then ab is an idempotent.

Proof:
(1) Suppose that a is group invertible with a group inverse b. Since a and b commute, we have that $a^kb^k = (ab)^k = (ba)^k = b^ka^k$, $b^ka^kb^k = (bab)^k = b^k$ and $a^kb^ka^k = (aba)^k = a^k$. We have thus shown that a^k is group invertible with a group inverse b^k.

(2) If $bab = b$, then $(ab)^2 = a(bab) = ab$, and hence ab is an idempotent. ■

Note that generalized inverses are not unique in general. For example, consider a regular element a and suppose that b is a regular inverse of a. One can then easily verify that the element bab is also a regular inverse of a. A logical question to ask is whether group inverses are unique provided they exist. The following result shows that this is indeed the case.

Proposition 3.1.4 *Let A be an algebra. A group invertible element in A can have at most one group inverse.*

Proof:
Let a be group invertible with group inverses b and c. Then, using Lemma 3.1.3(2) and the fact that both b and c commute with a, we have that

$$b = bab = b^2a = b^2aca = b^2a(ca)^2 = b^2ac^2a^2 = b^2a^3c^2$$

and

$$c = cac = ac^2 = abac^2 = (ab)^2ac^2 = a^2b^2ac^2 = b^2a^3c^2.$$

Hence $b = c$. ■

The group inverse of an algebra element a will be denoted by a^g.

Our next concern is the existence of a group inverse of an algebra element. We observe that, for an arbitrary algebra A, we do not have that $A = A^g$, that is, not every element is group invertible. For instance, if we consider the algebra $C[0, 1]$ and $a \in C[0, 1]$, defined by $a(x) = x$ for $x \in [0, 1]$, then this fact (see content of Example 3.2.2) can easily be seen from a result we will prove later on (see Lemma 3.2.1). This lemma describes the spectrum of a group invertible element.

Another example illustrating that not all algebra elements are group invertible is presented next.

Lemma 3.1.5 *Let A be an algebra. A non-zero element $a \in \mathrm{N}(A)$ cannot be group invertible.*

Proof:
Suppose this is not true, where $0 \neq a \in \mathrm{N}(A)$ is group invertible. Let k be the smallest positive integer such that $a^k = 0$. Since $a = aa^g a$, we have that $a^{k-2}a = a^{k-2}aa^g a$, that is, $a^{k-1} = a^k a^g = 0$. But this is a contradiction; hence the result follows. ■

For an algebra A with the property that $\mathrm{N}(A) \neq \{0\}$, Lemma 3.1.5 implies that $A \neq A^g$. An example of such an algebra is $M_2(\mathbb{C})$ (see content of Example 3.2.3).

In our next result we address the problem of developing conditions for the existence of a group inverse in an algebra.

Proposition 3.1.6 *([25], Lemma 3) Let A be an algebra. An element $a \in A$ is group invertible if and only if there exists an idempotent $p \in A$ such that*

$$ap = pa, \ \ a + p \textit{ is invertible and } ap = 0. \tag{3.1.1}$$

If the conditions in (3.1.1) are satisfied, then $a^g = (a + p)^{-1}(\mathbf{1} - p)$.

Proof:
Suppose that $a \in A$ is group invertible with $a^g = b$. Let $p := \mathbf{1} - ba$. By Lemma 3.1.3(2), p is an idempotent. We also have that $ap = pa = 0 = pb = bp$. It then follows that

$$(a + p)(b + p) = ab + ap + pb + p^2 = ab + p = \mathbf{1}.$$

Since $a+p$ and $b+p$ commute, we have that $a+p$ is invertible with $(a+p)^{-1} = b + p$.

Conversely, suppose that p is any idempotent satisfying the conditions in (3.1.1). We show that $b = (a + p)^{-1}(\mathbf{1} - p)$ is the group inverse of a: Since

$ap = pa$ we have that $ba = (a+p)^{-1}(\mathbf{1}-p)a = a(a+p)^{-1}(\mathbf{1}-p) = ab$. Using the fact that $p(\mathbf{1}-p) = 0$, it follows that $(a+p)(\mathbf{1}-p) = a(\mathbf{1}-p)$. Hence

$$\begin{aligned} bab &= [(a+p)^{-1}(\mathbf{1}-p)]a[(a+p)^{-1}(\mathbf{1}-p)] \\ &= (a+p)^{-1}(\mathbf{1}-p)(a+p)^{-1}a(\mathbf{1}-p) \\ &= (a+p)^{-2}(\mathbf{1}-p)a \\ &= [(a+p)^{-2}(\mathbf{1}-p)](a+p) \\ &= (a+p)^{-1}(\mathbf{1}-p) \\ &= b \end{aligned}$$

and

$$aba = a[(a+p)^{-1}(\mathbf{1}-p)]a = (a+p)[(a+p)^{-1}(\mathbf{1}-p)]a = (\mathbf{1}-p)a = a.$$

This completes the proof. ■

Corollary 3.1.7 *([25], Corollary 2) Let A be an algebra and $a \in A^g$. Then the idempotent p satisfying the conditions in (3.1.1) is unique.*

Proof:
Suppose that $a \in A^g$ and that p_1 and p_2 are idempotents in A satisfying the conditions in (3.1.1). By Proposition 3.1.6, both $(a+p_1)^{-1}(\mathbf{1}-p_1)$ and $(a+p_2)^{-1}(\mathbf{1}-p_2)$ are group inverses of a. It then follows from Proposition 3.1.4 that $(a+p_1)^{-1}(\mathbf{1}-p_1) = (a+p_2)^{-1}(\mathbf{1}-p_2)$, and hence $(\mathbf{1}-p_1)(a+p_2) = (a+p_1)(\mathbf{1}-p_2)$. This gives $a+p_2-ap_1-p_1p_2 = a-ap_2+p_1-p_1p_2$. Using the fact that $ap_1 = 0 = ap_2$, we have that $p_1 = p_2$, confirming the uniqueness. ■

We call the idempotent p in Proposition 3.1.6 the *group idempotent* of a.

In Corollary 3.1.8 we give the expression for the group idempotent of an algebra element.

Corollary 3.1.8 *Let A be an algebra and $a \in A^g$. Then the group idempotent of a is given by $p = \mathbf{1} - a^g a$.*

Proof:
This is clear by the proof of Proposition 3.1.6. ■

Let us remark that, by the symmetric property of group inverses, it follows from Corollary 3.1.8 that a and a^g have the same group idempotent.

We close this section by asking ourselves the following natural question: Is the sum of two group invertible elements group invertible? We answer this question in the next example.

Example 3.1.9 *Let A be an algebra and $a, b \in A^g$. The element $a+b$ is not group invertible in general.*

Consider the algebra $M_2(\mathbb{C})$. Let $a = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $b = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Since both a and b are invertible, they are group invertible. Let $c = a + b = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Then $c^2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Since c is a non-zero nilpotent element, we have from Lemma 3.1.5 that c is not group invertible. ■

The preceding example shows that the sum of two group invertible elements is not group invertible in general. The following lemma not only confirms the existence of the group inverse of the sum of two group invertible elements a and b under the condition $ab = ba = 0$, but also gives an expression for it.

Lemma 3.1.10 *Let A be an algebra and $a, b \in A^g$. If $ab = ba = 0$, then $a + b \in A^g$ with $(a + b)^g = a^g + b^g$.*

Proof:
Suppose that a and b are elements in A^g satisfying $ab = ba = 0$. Using the fact that $ab = ba = 0$, one can easily verify that $b^g \in \text{Comm}\ a$ where $ab^g = 0$ and $a^g \in \text{Comm}\ b$ where $a^g b = 0$. Hence $(a+b)(a^g+b^g) = aa^g+bb^g = (a^g+b^g)(a+b)$. We also have that

$$(a + b)(a^g + b^g)^2 = (a + b)((a^g)^2 + 2a^g b^g + (b^g)^2) = a(a^g)^2 + b(b^g)^2 = a^g + b^g$$

and

$$\begin{aligned} (a + b) - (a + b)^2(a^g + b^g) &= (a + b) - [(a^2 + 2ab + b^2)(a^g + b^g)] \\ &= a + b - (a^2 a^g + b^2 b^g) \\ &= a - a^2 a^g + b - b^2 b^g = 0, \end{aligned}$$

so that $a + b$ is group invertible with $(a + b)^g = a^g + b^g$. ■

3.2 The spectrum of a group invertible element

In this section we describe the spectrum of a group invertible element. We have the following result by Roch and Silbermann.

Lemma 3.2.1 *([25], Lemma 4) Let A be a Banach algebra and $a \in A^g$. Then either $0 \notin \sigma(a)$ (i.e. a is invertible) or $0 \in \text{iso}\,\sigma(a)$.*

Proof:
Suppose that $a \in A^g$. If $a = 0$, then $\sigma(a) = \{0\}$ and hence 0 is an isolated point of the spectrum. If $a \neq 0$ is group invertible with $a^g = b$, then $b \neq 0$. Let $\lambda \in \mathbb{C}$ be such that $0 < |\lambda| < \frac{1}{||b||}$, that is $||\lambda b|| < 1$. By Theorem 2.1.9 we have that $\mathbf{1} - \lambda b$ is invertible. Using simple algebraic arguments and the fact that b is the group inverse of a, we obtain that $\left[(\mathbf{1} - \lambda b)^{-1} b - \frac{1}{\lambda}(\mathbf{1} - ba)\right](a - \lambda \mathbf{1}) = \mathbf{1}$, so that $a - \lambda \mathbf{1}$ is invertible. Since $\lambda \notin \sigma(a)$ for all λ satisfying $0 < |\lambda| < \frac{1}{||b||}$, we have that $\sigma(a) \subseteq \{0\} \cup \{\lambda : |\lambda| \geq \frac{1}{||b||}\}$. The result then follows. ■

Recall that in the remark following Proposition 3.1.4, we stated that not all algebra elements are group invertible. Having formulated and proved Lemma 3.2.1, we now show that the element $a \in C[0,1]$, defined by $a(x) = x$ for $x \in [0,1]$, is not group invertible.

Example 3.2.2 *Let A be a Banach algebra. Then $A \neq A^g$ in general.*

For $a \in C[0,1]$, defined by $a(x) = x$ for all $x \in [0,1]$, we have that $\sigma(a) = a[0,1] = [0,1]$. Since $0 \in \sigma(a)\backslash\text{iso}\,\sigma(a)$, it follows from Lemma 3.2.1 that a is not group invertible. We have thus shown that the algebra $C[0,1]$ has the property that $C[0,1] \neq C[0,1]^g$. ■

A natural question would be if the converse of Lemma 3.2.1 is also true, that is, if a Banach algebra element a is such that $0 \in \text{iso}\,\sigma(a)$, whether we can conclude that a is group invertible? The answer is no in general and we provide the following example to illustrate this.

Example 3.2.3 *Let A be a Banach algebra and $a \in A$. Then $0 \in \text{iso}\,\sigma(a)$ does not imply that a is group invertible in general.*

Consider the algebra $M_2(\mathbb{C})$ and let $a = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Then $a^2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, that is, $a \in \mathrm{N}(M_2(\mathbb{C})) \subseteq \mathrm{QN}(M_2(\mathbb{C}))$, so that $\sigma(a) = \{0\}$. Hence, the element a has the property that $0 \in \text{iso}\,\sigma(a)$. However, by Lemma 3.1.5, $a \neq 0$ is not group invertible. ■

The following proposition shows that in the case of semisimple commutative Banach algebras, the spectral condition in Lemma 3.2.1 is sufficient, yielding a spectral characterization for the existence of a group inverse in a semisimple commutative Banach algebra. This result was proved in [25] by Roch and Silbermann. However, we will give a different proof using the HFC (Theorem 2.3.2). Note that the condition "$0 \notin \sigma(a)$ or $0 \in \text{iso}\,\sigma(a)$" is equivalent to $0 \notin \text{acc}\,\sigma(a)$.

Proposition 3.2.4 *([25], Proposition 2) Let A be a semisimple commutative Banach algebra. An element $a \in A$ is group invertible if and only if $0 \notin \text{acc}\,\sigma(a)$.*

Proof:
We prove only the sufficiency; the necessity follows from Lemma 3.2.1. To this end suppose that $0 \notin \text{acc}\,\sigma(a)$. Since $a \in A^{-1}$ implies that a is group invertible, we are only left to show that $0 \in \text{iso}\,\sigma(a)$ implies that a is group invertible.
Let $0 \in \text{iso}\,\sigma(a)$ and U_1 and U_0 be disjoint open neighbourhoods containing $\{0\}$ and $\sigma'(a)$ respectively. Then $U = U_0 \cup U_1$ is an open neighbourhood containing $\sigma(a)$. Define $f : U \to \mathbb{C}$ as follows:

$$f(\lambda) = \begin{cases} 0 & \text{if } \lambda \in U_0 \\ 1 & \text{if } \lambda \in U_1 \end{cases}$$

By Theorem 2.3.3, $p = f(a)$ is the spectral idempotent of a corresponding to 0. Moreover, $p \neq 0$ and commutes with a. Let $g(\lambda) = f(\lambda) + \lambda$. Then

$$g(\lambda) = \begin{cases} \lambda & \text{if } \lambda \in U_0 \\ 1 + \lambda & \text{if } \lambda \in U_1. \end{cases}$$

Moreover, $g \in H(U)$ and

$$g(\sigma(a)) = \{g(\lambda) : \lambda \in \sigma'(a)\} \cup \{g(\lambda) : \lambda = 0\} = \sigma'(a) \cup \{1\}.$$

Hence $0 \notin g(\sigma(a))$ and by the spectral mapping theorem (HFC(5)) it follows that $0 \notin \sigma(g(a))$, so that $p + a = f(a) + a = g(a) \in A^{-1}$.
Let $h(\lambda) = \lambda f(\lambda)$. Then

$$h(\lambda) = \begin{cases} 0 & \text{if } \lambda \in U_0 \\ \lambda & \text{if } \lambda \in U_1. \end{cases}$$

Moreover, $h \in H(U)$ and $h(a) = af(a) = ap$. Since

$$h(\sigma(a)) = h[(\sigma(a) \cap U_1) \cup (\sigma(a) \cap U_0)] = h(\{0\}) \cup \{0\} = \{0\},$$

we have again by the spectral mapping theorem that

$$\sigma(ap) = \sigma(h(a)) = h(\sigma(a)) = \{0\}.$$

Hence $ap \in \mathrm{QN}(A)$. Since A is semisimple, $\mathrm{Rad}(A) = \{0\}$, and by using the fact that A is also commutative, we have from Lemma 2.2.10 that $ap = 0$. By Proposition 3.1.6 we have that a is group invertible. ■

Corollary 3.2.5 *Let A be a semisimple commutative Banach algebra and $a \in A$. If $0 \in \operatorname{iso} \sigma(a)$, then the spectral idempotent of a corresponding to 0 is the group idempotent of a.*

Proof:
Suppose that $0 \in \operatorname{iso} \sigma(a)$ and let p denote the spectral idempotent of a corresponding to 0. By Proposition 3.2.4, we have that $a \in A^g$. By the proof of Proposition 3.2.4, p is an idempotent satisfying the conditions in (3.1.1). Hence the result follows. ■

Chapter 4

Drazin inverses

In his 1958 paper [9], Drazin introduced and studied the concept of a Drazin invertible element in associative rings and semigroups. The purpose of this chapter is to develop the theory of Drazin inverses in an algebra. This chapter consists of two sections. The aim of Section 4.1 is to define the concept of a Drazin inverse in an algebra and discuss several of its properties, such as the existence and uniqueness of the Drazin inverse. In Section 4.2 we present a spectral characterization for Drazin invertibility in semisimple commutative Banach algebras and develop some results from it. Let us mention that the group inverse investigated in Chapter 3 is a special case of the Drazin inverse. We will, however, establish a structure under which these concepts coincide. The main results in this chapter are Lemma 4.1.5, Proposition 4.1.9 and Lemma 4.2.1.

4.1 Introducing the Drazin inverse in algebras

Definition 4.1.1 (Drazin inverse) *([12], Lemma 2.1) Let A be an algebra. An element $a \in A$ is said to be Drazin invertible if there exists an element $b \in A$ such that:*

- $ab = ba$
- $bab = b$
- $a - aba \in \mathrm{N}(A)$

We call b in Definition 4.1.1 a *Drazin inverse* of a. We will see later (Lemma 4.1.5) that Drazin inverses are actually unique. It is easy to see from the definition of a Drazin inverse that every nilpotent element is Drazin invertible with a Drazin inverse 0. The element a in Definition 4.1.1 is said to be *Drazin invertible of degree k* if $(a-aba)^k = 0$. Recalling the definition of a group inverse, we see that group inverses are Drazin inverses of degree 1. Let A^d denote the set of Drazin invertible elements. Then the inclusions $A^{-1} \subsetneq A^g \subseteq A^d$ hold;

hence Drazin invertible elements, like group invertible elements, generalize invertible elements. We proceed by presenting an example which demonstrates a case where the inclusion $A^g \subseteq A^d$ is strict. Let us mention that a structure will be established later on (Corollary 4.2.2) in which the inclusion becomes an equality sign.

Example 4.1.2 *Let A be an algebra. The inclusion $A^g \subseteq A^d$ is strict in general.*

Consider the algebra $M_2(\mathbb{C})$. Let $a = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Then $a^2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Since a is nilpotent, it is Drazin invertible. By Lemma 3.1.5 $a \neq 0$ is not group invertible. Hence $M_2(\mathbb{C})^g \subsetneq M_2(\mathbb{C})^d$. ■

Note that, for an arbitrary algebra A, Lemma 3.1.5 implies that $a \in A^d \backslash A^g$ for all $0 \neq a \in \mathrm{N}(A)$.

Lemma 4.1.3 *Let A be an algebra and $a, b \in A$. If $b \in$ Comm a and $bab = b$, then for $k \in \mathbb{N}$, $(a - aba)^k = 0$ if and only if $a^k = a^k ba$.*

Proof:
Suppose that $b \in \mathrm{Comm}\ a$ and that $bab = b$. By Lemma 3.1.3(2) we have that ab is an idempotent and hence $\mathbf{1} - ab$ is also an idempotent. Let $k \in \mathbb{N}$. Then $a^k = a^k ba$ if and only if $0 = a^k(\mathbf{1} - ba) = a^k(\mathbf{1} - ba)^k = (a(\mathbf{1} - ba))^k = (a - aba)^k$. This completes the proof. ■

In almost all cases we will show that an algebra element $a - aba$ is nilpotent by showing that $a^k = a^k ba$ for some $k \in \mathbb{N}$, given that $ab = ba$ and $bab = b$.

Lemma 4.1.4 *([25], p.199) Let A be an algebra. Every Drazin inverse of degree k in A is a Drazin inverse of degree $k + 1$.*

Proof:
Suppose that $a \in A^d$ and let b be a Drazin inverse of a of degree k. In order to show that b is also a Drazin inverse of a of degree $k + 1$, we only need to prove that $a^{k+1} = a^{k+1} ba$. Since $a^k ba = a^k$, we have that $a^{k+1} = aa^k = a(a^k ba) = a^{k+1} ba$. Hence the result follows. ■

Our next result shows that Drazin inverses are unique if they exist.

Lemma 4.1.5 *([25], Lemma 1) Let A be an algebra. A Drazin invertible element $a \in A$ can have at most one Drazin inverse.*

Proof:
Suppose that $a \in A$ has a Drazin inverse b of degree k_1 and a Drazin inverse c of degree k_2, with $k_1 < k_2$. By Lemma 4.1.4 we have that b is also a Drazin inverse of a of degree k_2. Since both b and c satisfy the condition in Lemma

3.1.3(2), we have that both ab and ac are idempotents. Now, using the fact that both b and c are Drazin inverses of a of degree k_2, we have that

$$\begin{aligned} b &= bab \\ &= b(ab)^{k_2} \\ &= b^{k_2+1}a^{k_2} \\ &= b^{k_2+1}a^{k_2}ca \\ &= b^{k_2+1}a^{k_2}(ca)^{k_2+1} \\ &= b^{k_2+1}a^{2k_2+1}c^{k_2+1}. \end{aligned}$$

In a similar way one can show that $c = b^{k_2+1}a^{2k_2+1}c^{k_2+1}$, so that $c = b$. ■

The Drazin inverse of an algebra element a will be denoted by a^d.

The following lemma, due to Roch and Silbermann ([25]), gives a relation between the Drazin inverse and the group inverse. This result will be useful in Chapter 6.

Lemma 4.1.6 *([25], Lemma 2) Let A be an algebra. An element $a \in A$ is Drazin invertible of degree k in A if and only if a^k is group invertible in the algebra* Comm a*. Moreover, if a is Drazin invertible of degree k with $a^d = b$, then b^k is the group inverse of a^k, and if for some $k \in \mathbb{N}$, a^k is group invertible with $(a^k)^g = c$, then ca^{k-1} is the Drazin inverse of a of degree k, so that $b = b^k a^{k-1}$ and $c = (ca^{k-1})^k$.*

Proof:
Suppose that $a \in A$ is Drazin invertible of degree k with $a^d = b$. Then, recalling the definition of a Drazin inverse and using Lemma 3.1.3(2), we have that

$$a^k b^k = (ab)^k = (ba)^k = b^k a^k,$$

$$b^k = (bab)^k = b^k a^k b^k \text{ and}$$
$$a^k = a^k ba = a^k (ba)^k = a^k b^k a^k.$$

Moreover, since $ab = ba$, we have that $ab^k = b^k a$. It then follows that a^k is group invertible in Comm a with group inverse b^k.

Conversely, suppose that for some $k \in \mathbb{N}$, a^k is group invertible in Comm a with $(a^k)^g = c$. Then ca^{k-1} is the Drazin inverse of a of degree k: Using the fact that a and c commute, we have that

$$a(ca^{k-1}) = ca^k = (ca^{k-1})a,$$

$$(ca^{k-1})a(ca^{k-1}) = ca^k ca^{k-1} = ca^{k-1} \text{ and}$$
$$a^k(ca^{k-1})a = a^k ca^k = a^k.$$

This completes the proof. ■

Corollary 4.1.7 *Let A be an algebra. If $a \in A$ is Drazin invertible of degree k, then a^k is Drazin invertible of degree $p \geq 1$. Moreover, $(a^k)^d = (a^d)^k$.*

Proof:
Suppose that $a \in A$ is Drazin invertible of degree k. By Lemma 4.1.6 we have that a^k is group invertible, that is Drazin invertible of degree 1, with $(a^k)^d = (a^k)^g = (a^d)^k$. The result follows from Lemma 4.1.4. ■

Let us remark that Corollary 4.1.7 is an analogue of Lemma 3.1.3(1) for Drazin inverses.

Following the definition of the Drazin inverse of a square matrix (see Definition 2.12.1) we remarked that every element of the algebra $M_n(\mathbb{C})$ is Drazin invertible. This fact is, however, not true for arbitrary algebras, as illustrated in the following example.

Example 4.1.8 *Let A be an algebra. Then $A \neq A^d$ in general.*

Consider the algebra $C[0,1]$. Define $a \in C[0,1]$ by $a(x) = x$ for all $x \in [0,1]$. Clearly, $a^p(x) = x^p$, for all $p \geq 1$. We show that a is not Drazin invertible in $C[0,1]$. We do this by contradiction: Suppose that a is Drazin invertible of degree k with $a^d = b$. Since $a^k = a^k ba$, we have that $a^k(x) = (a^k ba)(x)$, so that $x^k = x^k b(x)x$. This fact forces b to take on the form $b(x) = \frac{1}{x}$, for $x \in (0,1]$; hence $\lim_{x \to 0^+} b(x) = \infty$. But this is absurd, since it implies that b is not continuous at 0. This completes the proof. ■

Now that we know that, in general, not all algebra elements are Drazin invertible, we present the next result which provides us with necessary and sufficient conditions for the existence of a Drazin inverse in an algebra.

Proposition 4.1.9 *([25], Proposition 1) Let A be an algebra. An element $a \in A$ is Drazin invertible of degree k in A if and only if there exists an idempotent $p \in A$ such that*

$$ap = pa, \; a + p \textit{ is invertible and } a^k p = 0. \tag{4.1.1}$$

If the conditions in (4.1.1) are satisfied, then $a^d = (a+p)^{-1}(\mathbf{1} - p)$.

Proof:
Suppose that $a \in A$ is Drazin invertible of degree k with $a^d = b$. By Lemma 4.1.6, a^k is group invertible in the algebra Comm a with group inverse b^k. Using Proposition 3.1.6 and Corollary 3.1.7, let $p \in$ Comm a be the group idempotent of a^k. By Corollary 3.1.8, $p = \mathbf{1} - b^k a^k$, and by also using Lemma 3.1.3(2) and the fact that a and b commute, we have that $p = \mathbf{1} - ba$. Moreover, $a^k p = pa^k$, $a^k + p$ is invertible and $a^k p = 0$. Since $p \in$ Comm a, we have that $ap = pa$. We are only left to show that $a + p$ is invertible. In order to do so, we show that $a^k + p$ is invertible if and only if $a + p$ is invertible:
By using induction and the fact that $ap = pa$, it can be shown that

$$a^n(\mathbf{1} - p) + p = [(\mathbf{1} - p)a(\mathbf{1} - p) + p]^n$$

for $n \geq 1$. Since $a^k p = 0$, it follows that

$$a^k + p = a^k(\mathbf{1} - p) + p = [(\mathbf{1} - p)a(\mathbf{1} - p) + p]^k,$$

and hence

$$a^k + p \text{ is invertible if and only if } (\mathbf{1} - p)a(\mathbf{1} - p) + p \text{ is invertible.} \qquad (4.1.2)$$

We also have that

$$\begin{aligned} a + p &= a - 2ap + ap + ap + p \\ &= a(\mathbf{1} - 2p + p) + ap + p \\ &= a(\mathbf{1} - p)^2 + ap^2 + p \\ &= (\mathbf{1} - p)a(\mathbf{1} - p) + pap + p \\ &= c + r, \end{aligned}$$

where $c := (\mathbf{1} - p)a(\mathbf{1} - p) + p$ and $r := pap$. Now,

$$r^k = (pap)^k = (ap)^k = a^k p^k = a^k p = 0,$$

so that r is nilpotent. From Lemma 2.1.11 it follows that $a + p$ is invertible if and only if $(\mathbf{1} - p)a(\mathbf{1} - p) + p$ is invertible. The result then follows from (4.1.2).

Conversely, suppose that $p \in A$ is an idempotent satisfying the conditions in (4.1.1). Let $b = (a + p)^{-1}(\mathbf{1} - p)$. Since $ap = pa$ we have that

$$ab = a[(a + p)^{-1}(\mathbf{1} - p)] = [(a + p)^{-1}(\mathbf{1} - p)]a = ba.$$

We also have that

$$\begin{aligned} bab &= [(a + p)^{-1}(\mathbf{1} - p)]a[(a + p)^{-1}(\mathbf{1} - p)] \\ &= (a + p)^{-1}(\mathbf{1} - p)(a + p)^{-1}a(\mathbf{1} - p) \\ &= (a + p)^{-2}(\mathbf{1} - p)a \\ &= [(a + p)^{-2}(\mathbf{1} - p)](a + p) \\ &= (a + p)^{-1}(\mathbf{1} - p) \\ &= b \end{aligned}$$

and

$$\begin{aligned} a^k ba &= a^k[(a + p)^{-1}(\mathbf{1} - p)]a \\ &= a^k[(a + p)^{-1}(\mathbf{1} - p)](a + p) \\ &= a^k[(a + p)^{-1}(a + p)](\mathbf{1} - p) \\ &= a^k. \end{aligned}$$

Hence $b = (a+p)^{-1}(\mathbf{1}-p)$ is the Drazin inverse of a of degree k. This completes the proof. ∎

Corollary 4.1.10 *Let A be an algebra and $a \in A^d$. Then the idempotent p satisfying (4.1.1) is given by $p = \mathbf{1} - a^d a$ and hence it is unique.*

Proof:
Suppose that $a \in A^d$ and let $p \in A$ be an idempotent satisfying the conditions in (4.1.1). By Proposition 4.1.9, $a^d = (a+p)^{-1}(\mathbf{1}-p)$, and hence $(a+p)a^d = \mathbf{1} - p$. By also using the facts that $pa^d = 0$ and that a and a^d commute, we have that $p = \mathbf{1} - a^d a$. ■

We call the idempotent p in Proposition 4.1.9 the *Drazin idempotent* of a.

Remark 4.1.11 *Take note that, for an algebra element a, the condition $a^k p = 0$ in (4.1.1) means that $ap \in \mathrm{N}(A)$.*

Remark 4.1.12 *Notice that, as we generalized the definition of group inverses to that of Drazin inverses by requiring the algebra element $a - aba$ to be nilpotent (Definition 4.1.1) instead of just zero (Definition 3.1.2), the characterization for the existence of a Drazin inverse in terms of idempotents in Proposition 4.1.9 differs from that of group inverses in Proposition 3.1.6 by also requiring the algebra element ap to be nilpotent instead of just zero.*

The following lemma is an analogue of Lemma 3.1.10 for Drazin inverses.

Lemma 4.1.13 *Let A be an algebra and $a, b \in A^d$. If $ab = ba = 0$, then $a + b \in A^d$ with $(a+b)^d = a^d + b^d$.*

Proof:
Suppose that a and b are elements in A^d satisfying $ab = ba = 0$. Using the fact that $ab = ba = 0$, one can easily verify that $b^d \in \mathrm{Comm}\, a$ where $ab^d = 0$ and $a^d \in \mathrm{Comm}\, b$ where $a^d b = 0$. Hence $(a+b)(a^d+b^d) = aa^d + bb^d = (a^d+b^d)(a+b)$. Now,

$$(a+b)(a^d+b^d)^2 = (a+b)((a^d)^2 + 2a^d b^d + (b^d)^2) = a(a^d)^2 + b(b^d)^2 = a^d + b^d$$

and since the elements $a - a^2 a^d$ and $b - b^2 b^d$ are nilpotent and commute, we have from Lemma 2.1.12 that

$$\begin{aligned}(a+b) - (a+b)^2(a^d+b^d) &= (a+b) - [(a^2 + 2ab + b^2)(a^d + b^d)] \\ &= a + b - (a^2 a^d + b^2 b^d) \\ &= a - a^2 a^d + b - b^2 b^d \in \mathrm{N}(A).\end{aligned}$$

Hence $a + b$ is Drazin invertible with $(a+b)^d = a^d + b^d$. ■

4.2 A spectral characterization for Drazin invertibility in semisimple commutative Banach algebras.

In this section we provide a result (Lemma 4.2.1), due to Roch and Silbermann in [25], which gives a spectral characterization for the existence of a Drazin inverse in the special case of a semisimple commutative Banach algebra. We also present a few consequences of this result.

Lemma 4.2.1 *([25], Lemma 6, p.204) Let A be a semisimple commutative Banach algebra. An element $a \in A$ is Drazin invertible if and only if* $0 \notin \operatorname{acc} \sigma(a)$.

Proof:
By Lemma 4.1.6, $a \in A$ is Drazin invertible of degree k if and only if a^k is group invertible in Comm a. Since A is commutative, we have that Comm $a = A$. It follows from Proposition 3.2.4 that the group invertibility of a^k is equivalent to $0 \notin \operatorname{acc} \sigma(a^k)$. By the spectral mapping theorem we have that $\sigma(a^k) = (\sigma(a))^k = \{\lambda^k : \lambda \in \sigma(a)\}$. The result then follows. ■

The following corollary is an immediate consequence of Proposition 3.2.4 and Lemma 4.2.1.

Corollary 4.2.2 *([25], Corollary 4, p.205) If A is a semisimple commutative Banach algebra, then every Drazin invertible element is group invertible.*

Recall that, for an arbitrary algebra A, we have that $A^g \subseteq A^d$. Example 4.1.2 illustrates a case where the inclusion is strict. Corollary 4.2.2 establishes a much stronger relation between the sets A^g and A^d in a semisimple commutative Banach algebra: the sets coincide.

Corollary 4.2.3 *If A is a semisimple commutative Banach algebra, $a \in A$ and* $0 \in \operatorname{iso} \sigma(a)$, *then the spectral idempotent of a corresponding to* 0 *is the Drazin idempotent of a*

Proof:
The result follows from Corollaries 3.1.8, 3.2.5 and 4.1.10, Lemma 4.2.1 and Corollary 4.2.2. ■

Chapter 5

Generalized Drazin inverses

In this chapter we focus on a generalization of the Drazin inverse introduced in Chapter 4. As a reference, we study the paper of Koliha (see [12]) who introduced and investigated the concept of a generalized Drazin inverse in a Banach algebra. This chapter is divided into four sections. In Section 5.1 we address logical issues such as when a generalized Drazin inverse exists, whether it is unique (given that it exists) and how it can be constructed. In Section 5.2 we present a few basic properties generalized Drazin invertible elements have. Section 5.3 is devoted to the decomposition of a generalized Drazin invertible element, while the aim of Section 5.4 is to present a relation between the spectrum of a generalized Drazin invertible element and the spectrum of its generalized Drazin inverse. The main results in this chapter are Corollaries 5.1.10 and 5.1.11, Theorem 5.1.16, Theorem 5.3.1 and its corollaries and Theorem 5.4.1.

5.1 Introducing the generalized Drazin inverse in Banach algebras

Definition 5.1.1 (Generalized Drazin inverse) *([12], Definition 2.3) Let A be a Banach algebra. An element $a \in A$ is said to be generalized Drazin invertible if there exists an element $b \in A$ such that:*

- $ab = ba$
- $bab = b$
- $a - aba \in \mathrm{QN}(A)$

We call b in Definition 5.1.1 a *generalized Drazin inverse* of a. Let A^D denote the set of generalized Drazin invertible elements. Observe that all the equations in Definition 5.1.1 hold when a is invertible (choose $b = a^{-1}$). Hence, generalized Drazin invertibility is another generalization of invertibility, that is, $A^{-1} \subseteq A^D$.

Since $\mathrm{N}(A) \subseteq \mathrm{QN}(A)$, every Drazin invertible element is generalized Drazin invertible, that is, $A^d \subseteq A^D$. Consequently, the set of inclusions $A^{-1} \subseteq A^g \subseteq A^d \subseteq A^D$ hold. It is also clear from the definition of a generalized Drazin inverse that every quasinilpotent element is generalized Drazin invertible with a generalized Drazin inverse 0.

In Proposition 3.1.4 and Lemma 4.1.5, respectively, we showed that group inverses and Drazin inverses are unique if they exist. A natural question would be whether we also have that generalized Drazin inverses are unique. This is indeed the case, and we will prove this statement in Corollary 5.1.8. In order to prove our next result we make use of the fact that the generalized Drazin inverse is unique if it exists.

Lemma 5.1.2 *Let A be a Banach algebra. An element $a \in \mathrm{QN}(A) \backslash \mathrm{N}(A)$ cannot be Drazin invertible.*

Proof:
Suppose that this is not true, where $a \in \mathrm{QN}(A) \backslash \mathrm{N}(A)$ is Drazin invertible with $a^d = b$. Since $a \in \mathrm{QN}(A)$, we have that a is generalized Drazin invertible with generalized Drazin inverse 0. By the uniqueness of the generalized Drazin inverse we must have that $b = 0$. Also, because $a - aba \in \mathrm{N}(A)$, we have that $a \in \mathrm{N}(A)$. This is a contradiction; hence the result follows. ■

Lemma 5.1.2 implies that if $a \in \mathrm{QN}(A) \backslash \mathrm{N}(A)$, then $a \in A^D \backslash A^d$. (We will see later (Corollary 5.1.11) that the elements a of the set $A^D \backslash A^d$ can be completely characterized, provided that $0 \in \mathrm{iso}\, \sigma(a)$). We thus conclude that, for any Banach algebra A with the property that $\mathrm{QN}(A) \backslash \mathrm{N}(A) \neq \emptyset$, the inclusion $A^d \subseteq A^D$ is strict. We have the following result in which we show that $\mathfrak{L}(l^1)$ is one such Banach algebra.

Example 5.1.3 *([12], Example 8.1) Let A be a Banach algebra. The inclusion $A^d \subseteq A^D$ is strict in general.*

Consider the Banach algebra $\mathfrak{L}(l^1)$. Let

$$B = (a_{ij}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Since $||B|| = \sup\{\sum_{i=0}^{\infty} |a_{ij}| : j \in \mathbb{N}\} < \infty$, we have that $B \in \mathfrak{L}(l^1)$, where

$$B(\xi_1, \xi_2, \ldots, \xi_n, \ldots) = \left(0, \xi_1, \frac{1}{2}\xi_2, \ldots, \frac{1}{n}\xi_n, \ldots\right).$$

By using the notation $B^2(\xi_1, \xi_2, \ldots, \xi_n, \ldots) = B(B(\xi_1, \xi_2, \ldots, \xi_n, \ldots))$, we get that

$$B^n(\xi_1, \xi_2, \ldots, \xi_n, \ldots) = \left(\underbrace{0, \ldots, 0}_{\text{n}}, \frac{1}{n!}\xi_1, \frac{1}{(n+1)!}\xi_2, \ldots, \frac{1}{(n+(n-1))!}\xi_n, \cdots \right),$$

that is,

$$B^n = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{n!} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{(n+1)!} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{(n+2)!} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{(n+3)!} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\0\\\vdots\\0\end{matrix}}\right\} n \text{ rows} \\ \\ \\ \\ \\ \\ \end{matrix}.$$

Hence $||B^n|| = \frac{1}{n!}$, so that $B^n \neq 0$ for all $n \in \mathbb{N}$. This shows that B is not nilpotent. By Theorem 2.2.3(iii), $r(B) = \lim_{n\to\infty} ||B^n||^{\frac{1}{n}} = \lim_{n\to\infty}(\frac{1}{n!})^{\frac{1}{n}} = 0$, that is, B is quasinilpotent and hence generalized Drazin invertible with a generalized Drazin inverse 0. Since $B \in \mathrm{QN}(\mathfrak{L}(l^1)) \backslash \mathrm{N}(\mathfrak{L}(l^1))$, we have from Lemma 5.1.2 that B is not Drazin invertible; hence $\mathfrak{L}(l^1)^d$ is strictly contained in $\mathfrak{L}(l^1)^D$. ■

We proceed by giving necessary and sufficient conditions for the existence of a generalized Drazin inverse in a Banach algebra.

Proposition 5.1.4 *([12], Lemma 2.4) Let A be a Banach algebra. An element $a \in A$ is generalized Drazin invertible in A if and only if there exists an idempotent $p \in A$ such that*

$$ap = pa, \ \ a + p \textit{ is invertible and } ap \in \mathrm{QN}(A). \tag{5.1.1}$$

If the conditions in (5.1.1) are satisfied, then $(a+p)^{-1}(\mathbf{1}-p)$ is a generalized Drazin inverse of a.

Proof:
Suppose that $a \in A$ is generalized Drazin invertible with a generalized Drazin inverse b. Let $p = \mathbf{1} - ba$. Then p is an idempotent which commutes with a and $ap \in \mathrm{QN}(A)$. We are only left to show that $a + p \in A^{-1}$. Since $pb = 0$, we have that

$$(a+p)(b+p) = ab+ap+pb+p = ab+ap+\mathbf{1}-ab = \mathbf{1}+ap = (b+p)(a+p).$$

Since $ap \in \mathrm{QN}(A)$, we have that $\mathbf{1}+ap \in A^{-1}$. Moreover, since $a+p$ and $b+p$ commute, it follows that $a+p \in A^{-1}$.

Conversely, suppose that $p \in A$ is any idempotent satisfying the conditions in (5.1.1). Let $b = (a+p)^{-1}(\mathbf{1}-p)$. Since $ap = pa$ we have that

$$ab = a[(a+p)^{-1}(\mathbf{1}-p)] = [(a+p)^{-1}(\mathbf{1}-p)]a = ba.$$

We also have that

$$\begin{aligned} bab &= [(a+p)^{-1}(\mathbf{1}-p)]a[(a+p)^{-1}(\mathbf{1}-p)] \\ &= (a+p)^{-1}(\mathbf{1}-p)(a+p)^{-1}a(\mathbf{1}-p) \\ &= (a+p)^{-2}(\mathbf{1}-p)a \\ &= [(a+p)^{-2}(\mathbf{1}-p)](a+p) \\ &= (a+p)^{-1}(\mathbf{1}-p) \\ &= b \end{aligned}$$

and

$$\begin{aligned} a - aba &= a - a[(a+p)^{-1}(\mathbf{1}-p)]a \\ &= a - a^2[(a+p)^{-1}(\mathbf{1}-p)] \\ &= a[\mathbf{1} - a(a+p)^{-1}(\mathbf{1}-p)] \\ &= a\left[\mathbf{1} - (a+p)[(a+p)^{-1}(\mathbf{1}-p)]\right] \\ &= ap \in \mathrm{QN}(A). \end{aligned}$$

Hence $a \in A^D$ and b is a generalized Drazin inverse of a. ■

Remark 5.1.5 *Notice that, as we generalized the definition of Drazin inverses to that of generalized Drazin inverses by requiring the Banach algebra element $a - aba$ to be quasinilpotent (Definition 5.1.1) instead of just nilpotent (Definition 4.1.1), the characterization for the existence of a generalized Drazin inverse in terms of idempotents in Proposition 5.1.4 differs from that of Drazin inverses in Proposition 4.1.9 by also requiring the element ap to be quasinilpotent instead of just nilpotent.*

The following result, due to Koliha [12], leads to a spectral characterization for the existence of a generalized Drazin inverse in a Banach algebra. Let us mention that this result plays a very important role in developing the theory of generalized Drazin inverses in a Banach algebra.

Theorem 5.1.6 *([12], Theorem 3.1) Let A be a Banach algebra and $a \in A$. Then $0 \notin \operatorname{acc}\sigma(a)$ if and only if there exists an idempotent $p \in A$ such that*

$$ap = pa, \ \ a + p \text{ is invertible and } ap \in \mathrm{QN}(A). \tag{5.1.2}$$

In particular, $0 \in \operatorname{iso}\sigma(a)$ if and only if $p \neq 0$.

Proof:
Suppose that $0 \notin \operatorname{acc} \sigma(a)$. Since $\sigma(a) = \operatorname{acc} \sigma(a) \cup \operatorname{iso} \sigma(a)$, we have that either $0 \notin \sigma(a)$ or $0 \in \operatorname{iso} \sigma(a)$. If $a \in A^{-1}$, then a is generalized Drazin invertible and by Proposition 5.1.4 there exists an idempotent $p \in A$ such that $ap = pa$, $a+p \in A^{-1}$ and $ap \in \mathrm{QN}(A)$. Since $ap \in \mathrm{QN}(A)$ and a^{-1} and ap commute, we have from Corollary 2.2.6 that $\sigma(p) = \sigma(a^{-1}pa) \subseteq \sigma(a^{-1})\sigma(ap) = \{0\}$, and hence $\sigma(p) = \{0\}$ by Theorem 2.2.3(ii), so that $p = 0$. Hence we have shown that, if $a \in A^{-1}$, then $p = 0$ is the only idempotent which satisfies (5.1.2).
If $0 \in \operatorname{iso} \sigma(a)$, then $\sigma(a) = \{0\} \cup \sigma'(a)$. Let U_1 and U_0 be disjoint open neighbourhoods of 0 and $\sigma'(a)$ respectively and let $U = U_1 \cup U_0$. Then U is an open neighbourhood of $\sigma(a)$. Define $f : U \to \mathbb{C}$ as follows:

$$f(\lambda) = \begin{cases} 0 & \text{if } \lambda \in U_0 \\ 1 & \text{if } \lambda \in U_1 \end{cases}$$

By Theorem 2.3.3, $p = f(a)$ is the spectral idempotent of a corresponding to 0. Moreover, $p \neq 0$ and commutes with a. Let $h(\lambda) = \lambda f(\lambda)$. Then

$$h(\lambda) = \begin{cases} 0 & \text{if } \lambda \in U_0 \\ \lambda & \text{if } \lambda \in U_1. \end{cases}$$

Moreover, $h \in H(U)$ and $h(a) = af(a) = ap$. Since

$$h(\sigma(a)) = h[(\sigma(a) \cap U_1) \cup (\sigma(a) \cap U_0)] = \{0\},$$

we have by the spectral mapping theorem that

$$\sigma(ap) = \sigma(h(a)) = h(\sigma(a)) = \{0\},$$

and hence $ap \in \mathrm{QN}(A)$. Let $g(\lambda) = f(\lambda) + \lambda$. Then

$$g(\lambda) = \begin{cases} \lambda & \text{if } \lambda \in U_0 \\ 1+\lambda & \text{if } \lambda \in U_1. \end{cases}$$

Moreover, $g \in H(U)$ and

$$g(\sigma(a)) = \{g(\lambda) : \lambda \in \sigma'(a)\} \cup \{g(\lambda) : \lambda = 0\} = \sigma'(a) \cup \{1\}.$$

Hence $0 \notin g(\sigma(a))$ and by the spectral mapping theorem it follows that $0 \notin \sigma(g(a))$, so that $p + a = f(a) + a = g(a) \in A^{-1}$. Hence we have shown that the spectral idempotent of a corresponding to 0 satisfies the conditions in (5.1.2).

For the reverse inclusion, suppose that $p \in A$ is any idempotent satisfying the conditions in (5.1.2). If $p = 0$, then $a + p = a \in A^{-1}$, so that $0 \notin \sigma(a)$, and hence $0 \notin \operatorname{acc} \sigma(a)$. Suppose $p \neq 0$ and that $0 \in \sigma(a)$. Let $\lambda \in \mathbb{C}$. Then

$$(\lambda\mathbf{1} - ap)p + (\lambda\mathbf{1} - (p+a))(\mathbf{1} - p) = \lambda p - ap + \lambda\mathbf{1} - p - a - \lambda p + p + ap = \lambda\mathbf{1} - a. \tag{5.1.3}$$

If $||[(p+a)-\lambda\mathbf{1}]-(p+a)|| = |\lambda| < ||(p+a)^{-1}||^{-1}$, then by Theorem 2.7.1 we have that $\lambda\mathbf{1}-(p+a)\in A^{-1}$; hence $\{\lambda : |\lambda| < ||(p+a)^{-1}||^{-1}\} \subseteq \rho(p+a)$. If $\lambda \neq 0$, then since $ap \in \mathrm{QN}(A)$, it follows that $\lambda\mathbf{1}-ap \in A^{-1}$. Hence, for $0 < |\lambda| < ||(p+a)^{-1}||^{-1}$, we have that both $\lambda\mathbf{1}-(p+a)$ and $\lambda\mathbf{1}-ap$ are invertible. It is easy to verify that equation (5.1.3) implies that

$$(\lambda\mathbf{1}-a)[(\lambda\mathbf{1}-ap)^{-1}p+(\lambda\mathbf{1}-(p+a))^{-1}(\mathbf{1}-p)] = \mathbf{1},$$

and hence

$$(\lambda\mathbf{1}-a)^{-1} = (\lambda\mathbf{1}-ap)^{-1}p+(\lambda\mathbf{1}-(p+a))^{-1}(\mathbf{1}-p), \tag{5.1.4}$$

so that $\lambda \notin \sigma(a)$ if $0 < |\lambda| < ||(p+a)^{-1}||^{-1}$. This shows that $\sigma(a) \subseteq \{0\} \cup \{\lambda : |\lambda| \geq ||(p+a)^{-1}||^{-1}\}$. Since $p \neq 0$ and $0 \in \sigma(a)$, we must have that $0 \in \mathrm{iso}\,\sigma(a)$; hence $0 \notin \mathrm{acc}\,\sigma(a)$. ■

Proposition 5.1.4 together with Theorem 5.1.6 enables us to get the following spectral characterization for the existence of a generalized Drazin inverse: A Banach algebra element a is generalized Drazin invertible if and only if $0 \notin \mathrm{acc}\,\sigma(a)$.

Corollary 5.1.7 *Let A be a Banach algebra. If $p \neq 0$ is an idempotent in A satisfying the conditions in (5.1.2), then p is the spectral idempotent of a corresponding to 0, and hence there is at most one idempotent p satisfying (5.1.2).*

Proof:
Suppose that $p \neq 0$ is an idempotent in A satisfying the conditions in (5.1.2). By Theorem 5.1.6, $0 \in \mathrm{iso}\,\sigma(a)$. Let f be defined as in the proof of Theorem 5.1.6 and let Γ_1 and Γ_0 be smooth contours included in U_1 and U_0 respectively, surrounding $\{0\}$ and $\sigma'(a)$ respectively, with $\Gamma = \Gamma_1 \cup \Gamma_0$. By Theorem 2.2.3(i) and the proof of Theorem 5.1.6, $\lambda \mapsto (\lambda\mathbf{1}-(p+a))^{-1}$ is analytic on $\{\lambda : |\lambda| < ||(p+a)^{-1}||^{-1}\}$, and hence on Γ_1, so that by Cauchy's theorem we have that

$$\int_{\Gamma_1} (\lambda\mathbf{1}-(a+p))^{-1} d\lambda = 0. \tag{5.1.5}$$

If $|\lambda| > ||ap||$, then by Theorem 2.1.9 we have that

$$\left(\mathbf{1}-\frac{1}{\lambda}ap\right)^{-1} = \sum_{n=0}^{\infty}\left(\frac{1}{\lambda}ap\right)^{n},$$

that is,

$$(\lambda\mathbf{1}-ap)^{-1} = \sum_{n=0}^{\infty}\frac{1}{\lambda^{n+1}}(ap)^{n},$$

where $\sum_{n=0}^{\infty}\frac{1}{\lambda^{n+1}}(ap)^{n}$ is the Laurent series of $\lambda \mapsto (\lambda\mathbf{1}-ap)^{-1}$ about the point 0 on the unbounded annulus $\{\lambda : |\lambda| > ||ap||\}$. Since $\sigma(ap) = \{0\}$, we have that $\lambda \mapsto (\lambda\mathbf{1} - ap)^{-1}$ is analytic on $\mathbb{C}\backslash\{0\}$ and hence on the unbounded annulus $\{\lambda : |\lambda| > 0\}$. Hence $(\lambda\mathbf{1} - ap)^{-1}$ has a Laurent series on $\{\lambda : |\lambda| > 0\}$. Since a Laurent series is unique, it follows that

$$(\lambda\mathbf{1} - ap)^{-1} = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}}(ap)^n \tag{5.1.6}$$

on $\{\lambda : |\lambda| > 0\}$. Now, using the HFC, (5.1.4), (5.1.5), (5.1.6) and the facts that p is an idempotent and $f(\lambda) = 0$, for all $\lambda \in \Gamma_0$, we have that

$$\begin{aligned}
f(a) &= \frac{1}{2\pi i}\int_{\Gamma} f(\lambda)(\lambda\mathbf{1} - a)^{-1}d\lambda \\
&= \frac{1}{2\pi i}\int_{\Gamma_1} f(\lambda)(\lambda\mathbf{1} - a)^{-1}d\lambda + \frac{1}{2\pi i}\int_{\Gamma_0} f(\lambda)(\lambda\mathbf{1} - a)^{-1}d\lambda \\
&= \frac{1}{2\pi i}\int_{\Gamma_1} (\lambda\mathbf{1} - a)^{-1}d\lambda \\
&= \frac{1}{2\pi i}\int_{\Gamma_1} (\lambda\mathbf{1} - ap)^{-1}p\,d\lambda + \frac{1}{2\pi i}\int_{\Gamma_1} (\lambda\mathbf{1} - (a+p))^{-1}(\mathbf{1} - p)d\lambda \\
&= \frac{1}{2\pi i}\int_{\Gamma_1} \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}}(ap)^n p\,d\lambda \\
&= p\sum_{n=0}^{\infty} a^n \frac{1}{2\pi i}\int_{\Gamma_1} \frac{1}{\lambda^{n+1}}d\lambda \\
&= p,
\end{aligned}$$

since

$$\frac{1}{2\pi i}\int_{\Gamma_1} \frac{1}{\lambda^{n+1}}d\lambda = \begin{cases} 0 & \text{if } n \geq 1 \\ 1 & \text{if } n = 0. \end{cases}$$

Hence p is the spectral idempotent of a corresponding to 0. ■

We call the idempotent p in Proposition 5.1.4 (or equivalently Theorem 5.1.6) the *generalized Drazin idempotent of* a.

Corollary 5.1.8 *Let A be a Banach algebra and $a \in A^D$ with a generalized Drazin inverse b. If $0 \in \operatorname{iso}\sigma(a)$, then the generalized Drazin idempotent of a is the spectral idempotent of a corresponding to 0. Also, the generalized Drazin idempotent of a is given by $p = \mathbf{1} - ba$. Moreover, the generalized Drazin inverse is unique and it is given by the expression $b = (a+p)^{-1}(\mathbf{1} - p)$.*

Proof:
Suppose that $a \in A^D$ with a generalized Drazin inverse b and generalized Drazin idempotent p. If $0 \in \operatorname{iso}\sigma(a)$, then by using Theorem 5.1.6 and Corollary 5.1.7, we have that p is the spectral idempotent of a corresponding to 0. Moreover, by the proof of Lemma 5.1.4, $\mathbf{1} - ba$ is an idempotent satisfying the conditions in (5.1.2). Hence, $p = \mathbf{1} - ba$.

Suppose that b_1 and b_2 are generalized Drazin inverses of a. By the first part of the proof we have that $\mathbf{1} - ab_1 = \mathbf{1} - ab_2$, that is, $ab_1 = ab_2$. Recalling the definition of the generalized Drazin inverse, we have that

$$b_1 = b_1ab_1 = b_1ab_2 = b_2ab_2 = b_2.$$

The expression of the generalized Drazin inverse follows from Proposition 5.1.4. ∎

The generalized Drazin inverse of a Banach algebra element a will be denoted by a^D.

From this point on, if it is clear from the context that $0 \in \operatorname{iso}\sigma(a)$, we shall write "spectral idempotent of a corresponding to 0" instead of "generalized Drazin idempotent".

Corollary 5.1.9 *Let A be a Banach algebra, $a \in A$ and $0 \in \operatorname{iso}\sigma(a)$. Then 0 is a pole of order k of $(\lambda\mathbf{1} - a)^{-1}$ if and only if $ap \in \mathrm{N}(A)$, where p denotes the spectral idempotent of a corresponding to 0 and k is the smallest positive integer such that $(ap)^k = 0$. In particular, 0 is a simple pole of $(\lambda\mathbf{1} - a)^{-1}$ if and only if $ap = 0$.*

Proof:
Suppose that $0 \in \operatorname{iso}\sigma(a)$ and let p denote the spectral idempotent of a corresponding to 0. Since p satisfies the conditions in (5.1.2), we have that (5.1.4) and (5.1.6) hold. Moreover, by the proof of Theorem 5.1.6, the resolvent $(\lambda\mathbf{1} - (a+p))^{-1}$ is analytic on the set $B = \{\lambda : |\lambda| < ||(p+a)^{-1}||^{-1}\}$, and hence has a Taylor series on B, say

$$(\lambda\mathbf{1} - (a+p))^{-1} = \sum_{n=0}^{\infty} b_n\lambda^n. \tag{5.1.7}$$

If $0 < |\lambda| < ||(p+a)^{-1}||^{-1}$, then by using (5.1.4), (5.1.6) and (5.1.7), we have that

$$\begin{aligned}(\lambda\mathbf{1} - a)^{-1} &= (\lambda\mathbf{1} - ap)^{-1}p + (\lambda\mathbf{1} - (p+a))^{-1}(\mathbf{1} - p) \\ &= \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}}(ap)^n + (\mathbf{1} - p)\sum_{n=0}^{\infty} b_n\lambda^n, \end{aligned} \tag{5.1.8}$$

where (5.1.8) represents the Laurent series of $\lambda \mapsto (\lambda\mathbf{1} - a)^{-1}$ about the point 0 on $\{\lambda : 0 < |\lambda| < ||(p+a)^{-1}||^{-1}\}$.

Now suppose that $ap \in \mathrm{N}(A)$, where k is the smallest positive integer such that $(ap)^k = 0$. Then $(ap)^{k-1} \neq 0$ and by (5.1.8) it follows that 0 is a pole of order k of $(\lambda\mathbf{1} - a)^{-1}$.

Conversely, suppose that 0 is a pole of order k of $(\lambda\mathbf{1} - a)^{-1}$. Again by (5.1.8) and the definition of a pole, we have that k is the smallest positive integer such that $(ap)^{k-1} \neq 0$; hence $(ap)^k = 0$, so that $ap \in \mathrm{N}(A)$. ■

From Corollary 5.1.9 we obtain the following two corollaries that allow us to distinguish between elements which are Drazin invertible and those which are generalized Drazin invertible but not Drazin invertible. The following result was observed by Koliha in [12].

Corollary 5.1.10 *([12], Note 3.4) Let A be a Banach algebra, $a \in A$ and $0 \in \operatorname{iso}\sigma(a)$. Then 0 is a pole of order k of $(\lambda\mathbf{1} - a)^{-1}$ if and only if a is Drazin invertible of degree k, where k is the smallest integer making the Drazin invertibility possible. In particular, 0 is a simple pole of $(\lambda\mathbf{1} - a)^{-1}$ if and only if a is group invertible.*

Proof:
Suppose that $0 \in \operatorname{iso}\sigma(a)$ and let p denote the spectral idempotent of a corresponding to 0. By the definition of p we have that $ap = pa$.

If 0 is a pole of order k of $(\lambda\mathbf{1} - a)^{-1}$, then $a^k p = (ap)^k = 0$ by Corollary 5.1.9. We also have by the proof of Theorem 5.1.6 that $a + p \in A^{-1}$. It then follows from Proposition 4.1.9 that a is Drazin invertible of degree k.

Conversely, suppose that k is the smallest integer such that a is Drazin invertible of degree k. By Proposition 4.1.9 there exists an idempotent p such that $ap = pa$, $a + p \in A^{-1}$ and $ap \in \mathrm{N}(A) \subseteq \mathrm{QN}(A)$. By Corollary 5.1.7, p is the spectral idempotent of a corresponding to 0. Since $(ap)^k = 0$, the result follows from Corollary 5.1.9. ■

The following corollary is not stated explicitly in [12], but follows directly from the comment below Theorem 5.1.6 and Corollary 5.1.10.

Corollary 5.1.11 *Let A be a Banach algebra, $a \in A$ and $0 \in \operatorname{iso}\sigma(a)$. Then 0 is an essential singularity of $(\lambda\mathbf{1} - a)^{-1}$ if and only if $a \in A^D \backslash A^d$.*

In the 1980's Harte introduced the concept of a quasipolar element in a Banach algebra. We define this concept next.

Definition 5.1.12 (Quasipolar) *([10], Definition 7.5.2) Let A be a Banach algebra. An element a is quasipolar if there exists an idempotent $q \in A$ such that*

$$aq = qa, \;\; q \in Aa \cap aA \text{ and } a(\mathbf{1} - q) \in \mathrm{QN}(A). \tag{5.1.9}$$

Harte showed in [10] that quasipolar elements have the following spectral property, which provides a link between the quasipolar elements and the generalized Drazin invertible elements of a Banach algebra. We will, however, not present the proof given by Harte in [10], but that of Koliha in [12], which uses Theorem 5.1.6.

Theorem 5.1.13 *([10],Theorem 9.7.6) Let A be a Banach algebra. An element $a \in A$ is quasipolar if and only if $0 \notin \operatorname{acc}\sigma(a)$.*

Proof:
Suppose that $a \in A$ is quasipolar. Then there exists an idempotent $q \in A$ satisfying the conditions in (5.1.9). Let $p = \mathbf{1} - q$. Then $ap = pa$ and $ap \in \mathrm{QN}(A)$. We show that $a + p \in A^{-1}$: Since $q \in Aa \cap aA$, there exist elements u, v in A such that $q = ua$ and $q = av$, that is, $\mathbf{1} - p = ua = av$. It then follows that

$$(a+p)(uav+p) = auav + ap + puav + p^2 = (\mathbf{1}-p)^2 + ap + p = \mathbf{1} + ap.$$

In a similar way one can show that $(uav+p)(a+p) = \mathbf{1}+ap$. Since $ap \in \mathrm{QN}(A)$, we have that $\mathbf{1} + ap \in A^{-1}$ and using the fact that the elements $uav + p$ and $a + p$ commute, it follows that $a + p \in A^{-1}$. By Theorem 5.1.6, $0 \notin \operatorname{acc} \sigma(a)$.

Conversely, suppose that $0 \notin \operatorname{acc} \sigma(a)$. If $a \in A^{-1}$, let $q = \mathbf{1}$. It is easy to verify that $\mathbf{1}$ satisfies the conditions in the definition of quasipolar elements; hence a is quasipolar. If $0 \in \operatorname{iso} \sigma(a)$, let p denote the spectral idempotent of a corresponding to 0 and let $q = \mathbf{1} - p$. Then q is an idempotent satisfying $aq = qa$ and $a(\mathbf{1} - q) = ap \in \mathrm{QN}(A)$. Since $(a + p)(\mathbf{1} - p) = a(\mathbf{1} - p)$, we have that

$$q = \mathbf{1} - p = (\mathbf{1} - p)(a + p)(a + p)^{-1} = a(\mathbf{1} - p)(a + p)^{-1} \in aA$$

and

$$q = \mathbf{1}-p = (a+p)^{-1}(a+p)(\mathbf{1}-p) = (a+p)^{-1}a(\mathbf{1}-p) = (a+p)^{-1}(\mathbf{1}-p)a \in Aa.$$

Hence a is quasipolar. ■

Corollary 5.1.14 *Let A be a Banach algebra, $a \in A$ and $0 \in \operatorname{iso} \sigma(a)$. The idempotent q in (5.1.9) is unique and it is given by $q = \mathbf{1} - p$, where p is the spectral idempotent of a corresponding to* 0.

The idempotent q is called the spectral idempotent of a corresponding to $\sigma'(a)$.

Remark 5.1.15 *The remark following Theorem 5.1.6 together with Theorem 5.1.13 provides a link between the set of quasipolar elements and the set of generalized Drazin invertible elements, namely that they coincide.*

The next theorem summarizes equivalent formulations for the existence of a generalized Drazin inverse.

Theorem 5.1.16 *([12],Theorem 4.2) Let A be a Banach algebra and $a \in A$. Then the following statements are equivalent:*

(i) $0 \notin \operatorname{acc} \sigma(a)$.
(ii) There is a unique idempotent $p \in A$, the generalized Drazin idempotent of a, satisfying the conditions in (5.1.2).
(iii) a has a unique generalized Drazin inverse given by $a^D = (a+p)^{-1}(\mathbf{1}-p)$.
(iv) a is quasipolar.

Proof:
The equivalence of (i) and (ii) follows from Theorem 5.1.6 and Corollary 5.1.7, while the equivalence of (i) and (iv) is obvious from Theorem 5.1.13. Statements (ii) and (iii) are equivalent by Proposition 5.1.4, Corollary 5.1.7 and Corollary 5.1.8. This completes the proof. ■

Remark 5.1.17 *In the case of generalized Drazin inverses we obtain the same result as that for Drazin inverses (Lemma 4.2.1) without requiring the Banach algebra to be commutative and semisimple.*

Corollary 5.1.18 *If A is a semisimple commutative Banach algebra, then the concepts of generalized Drazin invertibility, Drazin invertibility and group invertibility coincide.*

Proof:
The result follows from Proposition 3.2.4, Lemma 4.2.1 and Theorem 5.1.16. ■

Recall that, for an arbitrary Banach algebra A, we have that $A^g \subseteq A^d \subseteq A^D$. Examples 4.1.2 and 5.1.3 indicate that these inclusions may in general be strict. However, Corollary 5.1.18 shows that $A^g = A^d = A^D$ in the case where A is a semisimple commutative Banach algebra.

5.2 Further properties of generalized Drazin invertible elements

We devote this section to discussing some algebraic properties of generalized Drazin invertible elements that will be useful in the successive section.

Example 5.2.1 *Generalized Drazin inverses are not symmetric in general.*

Consider the Banach algebra $M_2(\mathbb{C})$. Let $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $a \in \mathrm{QN}(A)$, and hence generalized Drazin invertible with $a^D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. But $(a^D)^D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. ■

Our next result gives criteria for generalized Drazin inverses to be symmetric. The proof we give is slighty different to that of Koliha in [12].

Theorem 5.2.2 *([12], Theorem 5.3) Let A be a Banach algebra, $a \in A$ and $0 \in \operatorname{iso}\sigma(a)$. Then $(a^D)^D = a$ if and only if 0 is a simple pole of $(\lambda\mathbf{1} - a)^{-1}$. Hence $(a^D)^D = a$ if and only if $a \in A^g$.*

Proof:
Suppose that $0 \in \operatorname{iso}\sigma(a)$ and let p denote the spectral idempotent of a corresponding to 0.

Let a be generalized Drazin invertible with $a^D = b$ and let b be generalized Drazin invertible with $b^D = a$. By Corollary 5.1.8, $p = \mathbf{1} - ba$. Now

$$0 = b^D - b^D b b^D = a - aba = a - a(\mathbf{1} - p) = ap.$$

By Corollary 5.1.9, 0 is a simple pole of $(\lambda\mathbf{1} - a)^{-1}$.
Conversely, suppose that 0 is a simple pole of $(\lambda\mathbf{1} - a)^{-1}$. By Corollary 5.1.10, a is group invertible and hence generalized Drazin invertible with $a^D = a^g$. Using the fact that group inverses are symmetric, we have that $(a^D)^D = (a^g)^D = (a^g)^g = a$. The last sentence in the formulation of Theorem 5.2.2 follows from Corollary 5.1.10. ∎

In our next result we express the element $(a^D)^D$ in general.

Theorem 5.2.3 *([12], Theorem 5.4) Let A be a Banach algebra. Suppose that $a \in A$ is generalized Drazin invertible and that p is the generalized Drazin idempotent of a. Then $(a^D)^D = a(\mathbf{1} - p)$.*

Proof:
Suppose that $a \in A^D$ and that p is the generalized Drazin idempotent of a. If $a \in A^{-1}$, then $p = 0$ and the result obviously holds. If $0 \in \operatorname{iso}\sigma(a)$, then p is the spectral idempotent of a corresponding to 0. By Corollary 5.1.8, $p = \mathbf{1} - a^D a$. We show that $b = a(\mathbf{1} - p)$ is the generalized Drazin inverse of a^D : Using the fact that $ap = pa$, we have that $a^D b = a^D a(\mathbf{1} - p) = a(\mathbf{1} - p)a^D = ba^D$. We also have that

$$\begin{aligned}
ba^D b &= a(\mathbf{1} - p)a^D a(\mathbf{1} - p) \\
&= a(\mathbf{1} - p)a^D a(a^D a) \\
&= a(\mathbf{1} - p)(a^D a a^D)a \\
&= a(\mathbf{1} - p)a^D a \\
&= a(\mathbf{1} - p)(\mathbf{1} - p) \\
&= a(\mathbf{1} - p) = b
\end{aligned}$$

and

$$\begin{aligned}
a^D - a^D b a^D &= a^D - a^D a(\mathbf{1} - p)a^D \\
&= a^D - a^D a(a^D a)a^D \\
&= a^D - (a^D a a^D)a a^D \\
&= a^D - a^D a a^D \\
&= 0 \in \mathrm{QN}(A).
\end{aligned}$$

Hence $(a^D)^D = b = a(\mathbf{1} - p)$. This completes the proof. ∎

We now investigate whether Lemma 3.1.10 also holds for generalized Drazin inverses. We find that it is indeed the case. Theorem 5.2.4 implies that, if $0 \in \operatorname{iso}\sigma(a) \cap \operatorname{iso}\sigma(b)$ and $ab = ba = 0$, then $0 \in \operatorname{iso}\sigma(a + b)$. The result is due to Koliha ([12]).

Theorem 5.2.4 *([12], Theorem 5.7) Let A be a Banach algebra and $a, b \in A^D$. If $ab = ba = 0$, then $a + b \in A^D$ with $(a + b)^D = a^D + b^D$.*

Proof:
Suppose that a and b are elements in A^D satisfying $ab = ba = 0$. Using the fact that $ab = ba = 0$, one can easily verify that $b^D \in \text{Comm}\, a$ where $ab^D = 0$ and $a^D \in \text{Comm}\, b$ where $a^D b = 0$. Hence $(a + b)(a^D + b^D) = aa^D + bb^D = (a^D + b^D)(a + b)$. Now,

$$(a+b)(a^D+b^D)^2 = (a+b)((a^D)^2+2a^Db^D+(b^D)^2) = a(a^D)^2+b(b^D)^2 = a^D+b^D$$

and since the elements $a - a^2a^D$ and $b - b^2b^D$ are quasinilpotent and commute, we have from Theorem 2.2.3(ii) and Corollary 2.2.6 that

$$\begin{aligned}(a + b) - (a + b)^2(a^D + b^D) &= (a + b) - [(a^2 + 2ab + b^2)(a^D + b^D)] \\ &= a + b - (a^2a^D + b^2b^D) \\ &= a - a^2a^D + b - b^2b^D \in \text{QN}(A).\end{aligned}$$

Hence $a+b$ is generalized Drazin invertible with $(a+b)^D = a^D+b^D$. ■

5.3 The decomposition of a generalized Drazin invertible element

The next result, due to Koliha ([12]), enables us to decompose both a Drazin and a generalized Drazin invertible element as the sum of two elements satisfying certain properties. In Chapter 6 of this thesis we will have a look at some of these results in the Banach algebra of bounded linear operators.

Theorem 5.3.1 *([12], Theorem 6.4) Let A be a Banach algebra and $a \in A$. Then $0 \in \text{iso}\,\sigma(a)$ if and only if there exist $x, y \in A$ such that*

$$a = x + y, \ xy = 0 = yx, \ 0 \textit{ is a simple pole of } (\lambda\mathbf{1} - x)^{-1} \textit{ and } y \in \text{QN}(A). \tag{5.3.1}$$

Proof:
Suppose that $0 \in \text{iso}\,\sigma(a)$ and let p denote the spectral idempotent of a corresponding to 0. Let $x = a(\mathbf{1} - p)$ and $y = ap$. Then $a = x + y$, $xy = yx = 0$ and by Theorem 5.1.6, we have that $y \in \text{QN}(A)$. From Theorem 2.3.3 it follows that $\sigma(x) = \sigma(a(\mathbf{1} - p)) = \sigma'(a) \cup \{0\} = \sigma(a)$; hence $0 \in \text{iso}\,\sigma(x)$ by assumption. Also, since $xp = 0$, it follows from Corollary 5.1.9 that 0 is a simple pole of $(\lambda\mathbf{1} - x)^{-1}$.

Conversely, suppose that x and y are elements in A satisfying the conditions in (5.3.1). If $a \in A^{-1}$, then since $y \in \text{QN}(A)$, we have that $\mathbf{1} - a^{-1}y \in A^{-1}$, and hence $x = a - y \in A^{-1}$. This contradicts the fact that 0 is a simple pole of $(\lambda\mathbf{1}-x)^{-1}$. Therefore $0 \in \sigma(a)$. By Corollary 5.1.10, x is group invertible and hence generalized Drazin invertible. Since $y \in \mathrm{QN}(A)$, it is generalized Drazin invertible with generalized Drazin inverse 0. Since we also have that $xy = yx = 0$, it follows from Theorem 5.2.4 that $a = x + y \in A^D$, and hence $0 \notin \operatorname{acc} \sigma(a)$ by Theorem 5.1.16, so that $0 \in \operatorname{iso} \sigma(a)$. ■

By Corollary 5.1.10, x in Theorem 5.3.1 is a group invertible element.

Corollary 5.3.2 *Let A be a Banach algebra, $a \in A$ and $0 \in \operatorname{iso}\sigma(a)$. The decomposition in Theorem 5.3.1 is unique and is given by $x = a(\mathbf{1}-p) = aa^Da$ and $y = ap$, where p is the spectral idempotent of a corresponding to* 0.

Note that a^D exists by Theorem 5.1.16.

Proof:

Suppose that $0 \in \operatorname{iso}\sigma(a)$ and denote by p the spectral idempotent of a corresponding to 0. Using Theorem 5.3.1, let x and y be elements of A satisfying (5.3.1). We also have from Theorem 5.1.16 that a is generalized Drazin invertible, and hence $a^D = (x+y)^D = x^D$ by Theorem 5.2.4. Since 0 is a simple pole of $(\lambda\mathbf{1}-x)^{-1}$, it follows from Theorem 5.2.2 that $x = (x^D)^D$. Using Theorem 5.2.3, we have that $x = (x^D)^D = (a^D)^D = a(\mathbf{1}-p)$. Hence $y = a - x = a - a(\mathbf{1}-p) = ap$. This completes the proof. ■

Corollary 5.3.3 (Core-quasinilpotent decomposition) *([12], Corollary 6.5) Let A be a Banach algebra. An element a is generalized Drazin invertible if and only if it has a unique decomposition of the form $a = x + y$ where $x \in A^g$ and $y \in \mathrm{QN}(A)$ are such that $xy = yx = 0$. Then $a^D = x^g$.*

Proof:

Suppose that $a \in A$ is generalized Drazin invertible. By Theorem 5.1.16, $0 \notin \operatorname{acc} \sigma(a)$. If a is invertible, then $x = a$ and $y = 0$ satisfy $a = x + y$, with $x \in A^g$, $y \in \mathrm{QN}(A)$ and $xy = yx = 0$. It is obvious that $a^D = x^g$.

If $0 \in \operatorname{iso} \sigma(a)$, then the result follows from Theorem 5.3.1 and the comment following Theorem 5.3.1. Moreover, since $x, y \in A^D$ and $xy = yx = 0$, it follows from Theorem 5.2.4 that $a^D = (x+y)^D = x^D + y^D = x^D = x^g$.

Conversely, suppose that $a \in A$ is the sum of an element $x \in A^g$ and an element $y \in \mathrm{QN}(A)$ such that $xy = yx = 0$. Since both x and y are generalized Drazin invertible and $xy = yx = 0$, it follows from Theorem 5.2.4 that $a = x+y$ is generalized Drazin invertible with $a^D = (x+y)^D = x^D + y^D = x^g$. ■

The element x in Corollary 5.3.3 is called the *core* of a and the element y is called the *quasinilpotent part* of a. Throughout the rest of the thesis, $a^{(c)}$ and $a^{(q)}$ will denote the core and quasinilpotent part of a, respectively. Observe, by the proof of Theorem 5.3.1, that $\sigma(a^{(c)}) = \sigma(a)$, for all $a \in A^D$.

Corollary 5.3.4 (Core-nilpotent decomposition) *([12], Corollary 6.5) Let A be a Banach algebra. An element a is Drazin invertible if and only if it has* *a unique decomposition of the form* $a = x + y$ *where* $x \in A^g$ *and* $y \in \mathrm{N}(A)$ *are such that* $xy = yx = 0$*. Then* $a^d = x^g$*.*

Proof:
Suppose that $a \in A$ is Drazin invertible. If a is invertible, then $x = a$ and $y = 0$ satisfy $a = x + y$, with $x \in A^g$, $y \in \mathrm{N}(A)$ and $xy = yx = 0$. Hence $a^d = a^{-1} = x^{-1} = x^g$.

Suppose that $a \notin A^{-1}$. It follows from Theorem 5.1.16 that $0 \in \operatorname{iso} \sigma(a)$. Let p denote the spectral idempotent of a corresponding to 0. By Theorem 5.3.1 and Corollary 5.3.2, $a = x + y$, where $xy = yx = 0$, $x \in A^g$ and $y = ap \in \mathrm{QN}(A)$. We also have from Corollary 5.1.10 that 0 is a pole of $(\lambda\mathbf{1} - a)^{-1}$, and hence $y = ap \in \mathrm{N}(A)$ by Corollary 5.1.9. Since x and y are Drazin invertible and $xy = yx = 0$, we have from Lemma 4.1.13 that $a^d = (x + y)^d = x^d + y^d = x^g$.

Conversely, suppose that $a \in A$ is the sum of elements $x \in A^g$ and $y \in \mathrm{N}(A)$ such that $xy = yx = 0$. Since both x and y are Drazin invertible and $xy = yx = 0$, we have from Lemma 4.1.13 that $a = x + y$ is Drazin invertible with $a^d = (x + y)^d = x^d + y^d = x^g$. ■

The elements x and y in Corollary 5.3.4 are called the *core* of a and the *nilpotent part* of a, respectively. We use the notation $a^{(n)}$ to denote the nilpotent part of a.

Remark 5.3.5 *Let us mention that Corollary 5.3.4 is a generalization of the well-known core-nilpotent decomposition of a square matrix (Theorem 2.12.3) to arbitrary Banach algebra elements. In ([11], Theorem 5) King also showed this result for bounded linear operators.*

The symmetric property of the group inverse, together with Corollaries 5.3.2, 5.3.3 and 5.3.4, leads to the following remark.

Remark 5.3.6 *If an algebra (a Banach algebra) element* a *is Drazin (generalized Drazin) invertible, then* a^d (a^D) *is group invertible with group inverse given by* $a(\mathbf{1} - p)$*, where* p *denotes the Drazin (generalized Drazin) idempotent of* a*.*

5.4 Spectral properties of the generalized Drazin inverse

If a generalized inverse b of a has the property that its non-zero spectrum consists of the reciprocals of the non-zero spectral points of a, then b is called a *spectral inverse*. In [12] Koliha showed that a^D is a spectral inverse of a generalized Drazin invertible element a. The aim of this section is to present this result. Let us also mention that this property makes the generalized Drazin inverse useful in matrix theory and various applications of matrices.

Theorem 5.4.1 *([12], Theorem 4.4) Let A be a Banach algebra, $a \in A$ and $0 \in \operatorname{iso}\sigma(a)$. Let U_1 and U_0 be disjoint open neighbourhoods of 0 and $\sigma'(a)$ respectively. Choose U_1 such that it does not contain -1. Then $U = U_0 \cup U_1$ is an open neighbourhood containing $\sigma(a)$. Define $f \in H(U)$ by*

$$f(\lambda) = \begin{cases} 0 & \textit{if } \lambda \in U_1 \\ \dfrac{1}{\lambda} & \textit{if } \lambda \in U_0. \end{cases}$$

Then $a^D = f(a)$, so that $a^D \in Comm^2\, a$, and

$$\sigma(a^D) = \{0\} \cup \left\{ \frac{1}{\lambda} : \lambda \in \sigma'(a) \right\};$$

hence

$$\sigma'(a^D) = \left\{ \frac{1}{\lambda} : \lambda \in \sigma'(a) \right\}.$$

Proof:
Suppose that $0 \in \operatorname{iso}\sigma(a)$ and let p denote the spectral idempotent of a corresponding to 0. By Theorem 5.1.16, we have that $a \in A^D$ with $a^D = (a+p)^{-1}(\mathbf{1}-p)$. Also, let U_0, U_1, U and f satisfy the hypothesis in the formulation of Theorem 5.4.1. Define $g : U \to \mathbb{C}$ by

$$g(\lambda) = \begin{cases} 0 & \text{if } \lambda \in U_0 \\ 1 & \text{if } \lambda \in U_1. \end{cases}$$

Then $g \in H(U)$ and $p = g(a)$ by Theorem 2.3.3. Let $h(\lambda) = g(\lambda) + \lambda$. Then

$$h(\lambda) = \begin{cases} \lambda & \text{if } \lambda \in U_0 \\ 1+\lambda & \text{if } \lambda \in U_1. \end{cases}$$

Moreover, $h \in H(U)$ and since $-1 \notin U_1$ and $0 \notin U_0$ we have that $0 \notin h(U)$, and hence $0 \notin h(\sigma(a))$. It then follows from the spectral mapping theorem that $0 \notin \sigma(h(a))$, so that

$$p + a = g(a) + a = h(a) \in A^{-1}.$$

Also, because $0 \notin h(U)$, we have that $\frac{1}{h}$ is analytic on U. Define $s : U \to \mathbb{C}$ as follows:

$$\begin{aligned} s(\lambda) &= \frac{1}{h(\lambda)}(1 - g(\lambda)) \\ &= \frac{1}{\lambda + g(\lambda)}(1 - g(\lambda)) \end{aligned}$$

Then $s \in H(U)$. Moreover,

$$s(\lambda) = \begin{cases} 0 & \text{if } \lambda \in U_1 \\ \dfrac{1}{\lambda} & \text{if } \lambda \in U_0, \end{cases}$$

and hence $f(\lambda) = s(\lambda)$ for all $\lambda \in U$, so that

$$f(a) = s(a) = (a + g(a))^{-1}(\mathbf{1} - g(a)) = (a + p)^{-1}(\mathbf{1} - p) = a^D.$$

By the spectral mapping theorem we have that

$$\sigma(a^D) = \sigma(f(a)) = f(\sigma(a)) = \{0\} \cup \left\{ \frac{1}{\lambda} : \lambda \in \sigma'(a) \right\},$$

and hence $\sigma'(a^D) = \{\frac{1}{\lambda} : \lambda \in \sigma'(a)\}$. This completes the proof. ■

The following result relies on Theorem 5.4.1 and will be required in Chapter 6. It gives a relation between the distance between 0 and the non-zero spectrum of a generalized Drazin invertible element, and the spectral radius of its generalized Drazin inverse.

Corollary 5.4.2 *([13], Lemma 1.3) Let A be a Banach algebra and $a \in A^D$. If $r(a) > 0$, then $\mathrm{D}(0, \sigma'(a)) = r(a^D)^{-1}$.*

Proof:
Suppose that $a \in A^D$ and $r(a) > 0$. By Theorem 5.1.16, $0 \notin \operatorname{acc} \sigma(a)$. If $0 \notin \sigma(a)$, then $a^D = a^{-1}$. Let $f(\lambda) = \lambda^{-1}$. Then f is holomorphic on any neighbourhood of $\sigma(a)$ which does not contain 0. By the spectral mapping theorem we have that

$$\begin{aligned} \sigma(a^D) = \sigma(a^{-1}) = \sigma(f(a)) &= f(\sigma(a)) \\ &= \{f(\lambda) : \lambda \in \sigma(a)\} \\ &= \{\lambda^{-1} : \lambda \in \sigma(a)\}. \end{aligned}$$

Now

$$\begin{aligned} r(a^D) &= \sup\{|\lambda^{-1}| : \lambda \in \sigma(a)\} \\ &= (\inf\{|\lambda| : \lambda \in \sigma(a)\})^{-1} = \mathrm{D}(0, \sigma(a))^{-1}. \end{aligned}$$

Hence $\mathrm{D}(0, \sigma'(a)) = r(a^D)^{-1}$, since $\sigma(a) = \sigma'(a)$. This completes the proof for the case where a is invertible.
If $0 \in \operatorname{iso} \sigma(a)$, then by Theorem 5.4.1, $\sigma(a^D) = \{\lambda^{-1} : \lambda \in \sigma'(a)\} \cup \{0\}$. It then follows that

$$\begin{aligned} r(a^D) &= \sup\{|\lambda^{-1}| : \lambda \in \sigma'(a)\} \\ &= (\inf\{|\lambda| : \lambda \in \sigma'(a)\})^{-1} = \mathrm{D}(0, \sigma'(a))^{-1}, \end{aligned}$$

that is, $\mathrm{D}(0, \sigma'(a)) = r(a^D)^{-1}$. The proof is completed. ■

Chapter 6

Continuity of group, Drazin and generalized Drazin inversion

In the preceding chapters we discussed three types of generalized inverses, namely the group, Drazin and generalized Drazin inverse. It is the uniqueness of the group, Drazin and generalized Drazin inverse (see Proposition 3.1.4, Lemma 4.1.5 and Corollary 5.1.8) that led us to study the maps $a \mapsto a^g$, $a \mapsto a^d$, and $a \mapsto a^D$ on A^g, A^d and A^D, respectively.

From Lemmas 3.1.10 and 4.1.13 and Theorem 5.2.4 it is already clear that these maps are not linear in general. Another key aspect being investigated when studying maps, is the continuity of the map. Recall from Theorem 2.7.1 that inversion is continuous on the set of invertible elements. We devote this chapter to the study of the continuity properties of the group, Drazin and generalized Drazin inverse in a general Banach algebra.

This chapter is divided into three sections. In Section 6.1 we introduce, for a subalgebra of a Banach algebra, the notion of inverse closedness with respect to regularity (group invertibility, Drazin invertibility). Various results obtained from this notion will be required in Section 6.2.

The aim of Section 6.2 is to investigate whether certain continuity results available for invertibility (see Section 2.7) also hold for group, Drazin and generalized Drazin invertibility. Finally, we present criteria for the continuity of generalized Drazin inversion in Banach algebras.

In Section 6.3 we present continuity properties of group and Drazin inversion of socle elements in the semisimple Banach algebra setting. In particular, we will obtain a generalization of Campbell and Meyer's continuity result for matrices (Theorem 2.12.4) to arbitrary socle elements of a semisimple Banach algebra.

The main results in this chapter are Proposition 6.2.3, Corollary 6.2.4, Theorems 6.2.13 and 6.3.5 and Corollary 6.3.6.

6.1 Inverse closedness

In this section we briefly discuss the concept of inverse closedness with respect to several kinds of generalized invertibility. The main results in this section are Proposition 6.1.2 and Corollary 6.1.4.

Recall that a subalgebra B of a Banach algebra A is said to be inverse closed in A if B is inverse closed with respect to invertibility, that is, if B contains the inverses of all its invertible elements (see Definition 2.2.12). Similarly, we have that a subalgebra B of a Banach algebra A is *inverse closed with respect to regularity (group invertibility, Drazin invertibility)* if B contains the regular (group, Drazin) inverses of all its regular (group invertible, Drazin invertible) elements.

Example 6.1.1 *([25], p. 205) Inverse closedness (with respect to invertibility) does not imply inverse closedness with respect to generalized invertibilty.*

Consider the Banach algebra $A = M_2(\mathbb{C})$. We use the notion of a regular inverse to illustrate this fact. Let

$$B = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}.$$

Then B is a closed subalgebra of A. If $c \in B^{-1}$, then $c = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$ for some $0 \neq \alpha, \beta \in \mathbb{C}$, with

$$c^{-1} = \frac{1}{\alpha^2} \begin{pmatrix} \alpha & -\beta \\ 0 & \alpha \end{pmatrix} \in B.$$

Hence B is inverse closed (with respect to invertibility). Consider the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in B$. Then $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \in M_2(\mathbb{C})$ is a regular inverse of $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Suppose that $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is regular in B and that, for some $\alpha, \beta \in \mathbb{C}$, $\left(\begin{smallmatrix} \alpha & \beta \\ 0 & \alpha \end{smallmatrix}\right) \in B$ is a regular inverse of $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. By the definition of regular elements we have that $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} \alpha & \beta \\ 0 & \alpha \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, that is $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. But this is absurd; hence B is not inverse closed with respect to regularity. ■

The following result provides conditions for a subalgebra to be inverse closed with respect to group invertibility.

Proposition 6.1.2 *([25], Proposition 3) Let A be a Banach algebra and B a closed subalgebra of A which contains the identity and which is inverse closed in A. Then B is inverse closed in A with respect to group invertibility.*

Proof:
Suppose that B is a closed subalgebra of A which contains the identity and which is inverse closed in A. Let $a \in B^g$ with group idempotent p and $a^g = (a+p)^{-1}(\mathbf{1}-p) \in A$. We are required to prove that $a^g \in B$. Since B is a closed and inverse closed subalgebra that contains the identity and a,

we have from Lemma 2.2.13 that $A[a] \subseteq B$. Since a is group invertible, it follows from Lemma 3.2.1 that either $0 \notin \sigma(a)$ or $0 \in \operatorname{iso}\sigma(a)$. If $0 \notin \sigma(a)$, then a is invertible and hence $a^g = a^{-1} \in B$ by assumption. If $0 \in \operatorname{iso}\sigma(a)$, then $\frac{1}{n} \notin \sigma(a)$, that is, $a - \frac{1}{n}\mathbf{1}$ is invertible for all sufficiently large n. It then follows that $a(a - \frac{1}{n}\mathbf{1})^{-1} \in A[a]$ for all sufficiently large n. Now, $\lim_{n\to\infty} \left[(\mathbf{1}-p)a(\mathbf{1}-p) + p - \frac{1}{n}(\mathbf{1}-p)\right] = (\mathbf{1}-p)a(\mathbf{1}-p) + p = a + p$. Since $a + p \in A^{-1}$ by Proposition 3.1.6, it follows from Lemma 2.7.3 that

$$(\mathbf{1}-p)a(\mathbf{1}-p) + p - \frac{1}{n}(\mathbf{1}-p) \in A^{-1}$$

for all sufficiently large n. Using the fact that $p(\mathbf{1}-p) = 0$, we have that

$$\left[(\mathbf{1}-p) - \tfrac{1}{n}p\right]\left[(\mathbf{1}-p) - np\right] = \mathbf{1},$$

and together with the fact that $ap = pa = 0$, it follows, for all sufficiently large n, that

$$\begin{aligned}
a\left(a - \frac{1}{n}\mathbf{1}\right)^{-1} &= a\left(a(\mathbf{1}-p) - \frac{1}{n}(\mathbf{1}-p+p)\right)^{-1} \\
&= a\left(a(\mathbf{1}-p)^2 - \frac{1}{n}(\mathbf{1}-p+p)\right)^{-1} \\
&= a\left((\mathbf{1}-p)a(\mathbf{1}-p) - \frac{1}{n}(\mathbf{1}-p) - \frac{1}{n}p\right)^{-1} \\
&= a\left(\left[(\mathbf{1}-p)a(\mathbf{1}-p) + p - \frac{1}{n}(\mathbf{1}-p)\right]\left[(\mathbf{1}-p) - \frac{1}{n}p\right]\right)^{-1} \\
&= a\left[(\mathbf{1}-p) - \frac{1}{n}p\right]^{-1}\left[(\mathbf{1}-p)a(\mathbf{1}-p) + p - \frac{1}{n}(\mathbf{1}-p)\right]^{-1} \\
&= a[(\mathbf{1}-p) - np]\left[(\mathbf{1}-p)a(\mathbf{1}-p) + p - \frac{1}{n}(\mathbf{1}-p)\right]^{-1} \\
&= a\left[(\mathbf{1}-p)a(\mathbf{1}-p) + p - \frac{1}{n}(\mathbf{1}-p)\right]^{-1}.
\end{aligned}$$

Hence, by using the fact that inversion is continuous on the set of invertible elements (Theorem 2.7.1), Corollary 3.1.8 and the expression of a^g, we have that

$$\begin{aligned}
\lim_{n\to\infty} a\left(a - \frac{1}{n}\mathbf{1}\right)^{-1} &= \lim_{n\to\infty} a\left[(\mathbf{1}-p)a(\mathbf{1}-p) + p - \frac{1}{n}(\mathbf{1}-p)\right]^{-1} \\
&= a\lim_{n\to\infty}\left[(\mathbf{1}-p)a(\mathbf{1}-p) + p - \frac{1}{n}(\mathbf{1}-p)\right]^{-1} \\
&= a(a+p)^{-1} \\
&= a(a^g + p) \\
&= aa^g \\
&= \mathbf{1} - p.
\end{aligned}$$

Since $a(a-\frac{1}{n}\mathbf{1})^{-1} \in A[a]$ for all sufficiently large n, and $A[a]$ is closed, we have that $\mathbf{1}-p \in A[a]$. Hence $p = -(\mathbf{1}-p)+\mathbf{1} \in A[a]$.
It then follows that $a+p \in A[a]$ and since $A[a]$ is inverse closed by Lemma 2.2.13, we have that $(a+p)^{-1} \in A[a]$. Hence $a^g = (a+p)^{-1}(\mathbf{1}-p) \in A[a] \subseteq B$.
■

Observe from Lemma 2.2.13, together with Proposition 6.1.2, that $A[a]$ is an example of a subalgebra which is inverse closed in A with respect to group invertibility.

Corollaries 6.1.3 and 6.1.4 are consequences of Proposition 6.1.2. Using the concept of inverse closedness with respect to group invertibility, Roch and Silbermann showed in [25] that, in the case where A is a Banach algebra, we can replace the group invertibility of the element a^k in Comm a (Lemma 4.1.6) with group invertibility in the algebra A itself. We formulate and prove this result next.

Corollary 6.1.3 *([25], Corollary 5) Let A be a Banach algebra. An element $a \in A$ is Drazin invertible of degree k in A if and only if a^k is group invertible in A.*

Proof:
In Lemma 4.1.6 we proved that if a is Drazin invertible of degree k in A, then a^k is group invertible in Comm a and hence in A. We are only left to show that group invertibility of a^k in A implies group invertibility of a^k in Comm a. Suppose that a^k is group invertible in A. Since $a^k \in$ Comm a and Comm a is a closed subalgebra containing the identity and a, and which is inverse closed in A, we have by Proposition 6.1.2 that Comm a is inverse closed in A with respect to group invertiblity. Hence a^k is group invertible in Comm a. By Lemma 4.1.6, a is Drazin invertible of degree k in A. The proof is completed.
■

Corollary 6.1.4 is an analogue of Proposition 6.1.2 for Drazin inverses and will be required in Section 6.2.

Corollary 6.1.4 *([25], Corollary 6) Let A be a Banach algebra and B a closed subalgebra of A which contains the identity and which is inverse closed in A. Then B is inverse closed in A with respect to Drazin invertibility.*

Proof:
Suppose that B is a closed subalgebra of A which contains the identity and which is inverse closed in A. By Lemma 2.2.13, $A[a] \subseteq B$. Let $a \in B^d$ and suppose that a is Drazin invertible of degree k. We are required to prove that $a^d \in B$. Since a is Drazin invertible of degree k in A, we have from Corollary 6.1.3 that a^k is group invertible in A. Since $a^k \in A[a]$, it follows from Lemma 2.2.13 and Proposition 6.1.2 that a^k is group invertible in $A[a]$. Let $b = (a^k)^g \in A[a]$. By Lemma 4.1.6, ba^{k-1} is the Drazin inverse of a of

degree k. Moreover, $a^d = ba^{k-1} \in A[a] \subseteq B$. Hence B is inverse closed in A with respect to Drazin invertibility. ■

Remark 6.1.5 *If a subalgebra B of a Banach algebra A is closed and contains the identity, then B being inverse closed with respect to invertibility implies that B is inverse closed with respect to both group invertibility (Proposition 6.1.2) and Drazin invertibility (Corollary 6.1.4).*

Definition 6.1.6 *([25], p.208) Let A be a Banach algebra. We define the following two sets of sequences in A :*

$$\ell^\infty(A) = \{(a_n) : a_n \in A, \sup\{||a_n|| : n \in \mathbb{N}\} < \infty\}$$
$$c(A) = \{(a_n) : a_n \in A, \lim_{n\to\infty} a_n \text{ exists}\}$$

The sets $\ell^\infty(A)$ and $c(A)$ (with elementwise operations and the supremum norm) are Banach algebras. It is obvious that $c(A) \subseteq \ell^\infty(A)$.

The following result will be required in the proof of Corollary 6.2.4.

Lemma 6.1.7 *Let A be a Banach algebra. The subalgebra $c(A)$ is inverse closed in $\ell^\infty(A)$.*

Proof:
Suppose that $(a_n) \in c(A)^{-1}$ with inverse $(a_n^{-1}) \in \ell^\infty(A)$. We are required to prove that $(a_n^{-1}) \in c(A)$. By the definition of $c(A)$, let a be the limit of (a_n). Since $(a_n^{-1}) \in \ell^\infty(A)$, we have that $\sup\{||a_n^{-1}|| : n \in \mathbb{N}\} < \infty$. From Lemma 2.7.2 it then follows that $a \in A^{-1}$ and $\lim_{n\to\infty} a_n^{-1} = a^{-1}$. Hence$(a_n^{-1}) \in c(A)$, so that $c(A)$ is inverse closed in $\ell^\infty(A)$. ■

6.2 Continuity results for group, Drazin and generalized Drazin invertibility in Banach algebras

The objective of this section is to present several continuity properties of the group, Drazin and generalized Drazin inverse of elements in a Banach algebra. We will, in particular, investigate whether certain standard continuity results for invertibility, presented in Section 2.7, also hold for group, Drazin and generalized Drazin invertibility.

It is well-known that inversion is continuous on A^{-1} (Theorem 2.7.1). A natural question would be whether group, Drazin and generalized Drazin inversion is continuous on A^g, A^d and A^D respectively. This is, however, not true in general and we demonstrate this in the following example.

Example 6.2.1 *Group (Drazin, generalized Drazin) inversion in Banach algebras is not continuous in general.*

Consider the Banach algebra $M_2(\mathbb{C})$. For $n \in \mathbb{N}$, let

$$a_n = \begin{pmatrix} 1 & \frac{1}{n}+1 \\ 0 & \frac{1}{n} \end{pmatrix}$$

and

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Since $\det(a_n) \neq 0$, we have that all a_n are invertible and hence group (Drazin, generalized Drazin) invertible with

$$a_n^g = a_n^d = a_n^D = a_n^{-1} = \begin{pmatrix} 1 & -(1+n) \\ 0 & n \end{pmatrix}.$$

We also have that, since a is an idempotent, a is group (hence Drazin and generalized Drazin) invertible with itself as group (Drazin, generalized Drazin) inverse. It is clear that $a_n \to a$ as $n \to \infty$, but the sequence of group (Drazin, generalized Drazin) inverses of a_n does not converge to the group (Drazin, generalized Drazin) inverse of a, in fact, the sequence of group (Drazin, generalized Drazin) inverses of a_n diverges. ■

We give another example illustrating the discontinuity of generalized Drazin inversion in Banach algebras.

Example 6.2.2 *([13], Example 2.2) Let A be a Banach algebra. Generalized Drazin inversion is not continuous on A^D in general.*

Suppose that $0 \neq a \in \mathrm{N}(A)$ and that 3 is the smallest integer such that $a^3 = 0$. Then a is Drazin invertible and hence generalized Drazin invertible with $a^D = 0$. Let $a_n = a + \frac{1}{n}$, for all $n \in \mathbb{N}$. Then, for each n, $\left(a + \frac{1}{n}\right)(n\mathbf{1} - n^2a + n^3a^2) = \mathbf{1}$, so that a_n is invertible and hence generalized Drazin invertible with

$$a_n^D = a_n^{-1} = n\mathbf{1} - n^2a + n^3a^2.$$

It is clear that $a_n \to a$ as $n \to \infty$, but (a_n^D) does not converge to a^D, since (a_n^D) is unbounded. ■

Next, we investigate whether an analogous result of Lemma 2.7.2 is possible for group, Drazin and generalized Drazin inversion in Banach algebras. In [25] Roch and Silbermann obtained the following two results, one dealing with group invertibility (Proposition 6.2.3) and the other with Drazin invertibility (Corollary 6.2.4). We omit the proof of the following proposition, whose proof is given in [25]. We will, however, present the proof of Corollary 6.2.4, which is not in [25]. Let us also mention that the proofs of Proposition 6.2.3 and Corollary 6.2.4 are very similar.

Proposition 6.2.3 *([25], Proposition 4) Let A be a Banach algebra, (a_n) a convergent sequence in A with limit a, and suppose all a_n are group invertible. Then the following statements are equivalent:*
(a) $\sup\{||a_n^g|| : n \in \mathbb{N}\} < \infty$.
(b) The sequence (a_n^g) is convergent (say with limit b) and a is group invertible with $a^g = b$.

Let a_n be Drazin invertible of degree k_n, where k_n is the smallest positive integer for which a_n is Drazin invertible. We say that the *Drazin degrees* (k_n) *are uniformly bounded* if there exists $c > 0$ such that $k_n \leq c$ for all $n \in \mathbb{N}$.

Corollary 6.2.4 *([25], Corollary 7) Let A be a Banach algebra, (a_n) a convergent sequence in A with limit a, and suppose all a_n are Drazin invertible of uniformly bounded Drazin degree. Then the following statements are equivalent:*
(a) $\sup\{||a_n^d|| : n \in \mathbb{N}\} < \infty$.
(b) The sequence (a_n^d) is convergent (say with limit b) and a is Drazin invertible of degree $\sup\{k_n : n \in \mathbb{N}\}$ *with $a^d = b$.*

Proof:
Suppose that $a_n \to a$ as $n \to \infty$, where all a_n are Drazin invertible of degree k_n and k_n is the smallest positive integer for which a_n is Drazin invertible. Assume also that the Drazin degrees (k_n) are uniformly bounded. For the non-trivial implication, suppose that (a) holds. Let $k = \sup\{k_n : n \in \mathbb{N}\}$. By Lemma 4.1.4, all a_n are Drazin invertible of degree k. Moreover, $(a_n^d) \in \ell^\infty(A)$ where (a_n^d) is the Drazin inverse of (a_n) of degree k. But (a_n) is an element of the algebra $c(A)$ which is inverse closed in $\ell^\infty(A)$ by Lemma 6.1.7. It follows from Corollary 6.1.4 that $c(A)$ is inverse closed in $\ell^\infty(A)$ with respect to Drazin invertibility. Hence $(a_n^d) \in c(A)$, that is, the sequence (a_n^d) is convergent. Let b be the limit of (a_n^d). Now, by the continuity of multiplication in a Banach algebra, we have that

$$a^k ba = \lim_{n\to\infty} (a_n^k a_n^d a_n) = \lim_{n\to\infty} a_n^k = a^k,$$

$$bab = \lim_{n\to\infty} (a_n^d a_n a_n^d) = \lim_{n\to\infty} a_n^d = b,$$

and $ab = \lim_{n\to\infty}(a_n a_n^d) = \lim_{n\to\infty}(a_n^d a_n) = ba$. Hence a is Drazin invertible of degree $\sup\{k_n : n \in \mathbb{N}\}$ with Drazin inverse b. ■

Remark 6.2.5 *Proposition 6.2.3 and Corollary 6.2.4 (assuming that the Drazin degrees are uniformly bounded) show that Lemma 2.7.2 also holds when replacing invertibility with group and Drazin invertibility, respectively.*

In [13], Koliha and Rakočević presented the following example which illustrates that Lemma 2.7.2 does not, in general, hold when replacing invertibility with generalized Drazin invertibilty.

Example 6.2.6 *([13], Example 2.3, [2], p.49) Lemma 2.7.2 does not hold for generalized Drazin invertibility in Banach algebras in general.*

Proof:
Consider the Banach algebra $\mathfrak{L}(l^2)$ and denote by $e_1, e_2, \ldots$ the standard Schauder basis for l^2. Let (α_n) be the sequence of positive numbers defined by $\alpha_n = e^{-k_n}$, where $n = 2^{k_n}(2l_n + 1)$. Define $T : l^2 \to l^2$ by $Te_n = \alpha_n e_{n+1}$. Then T is the unilateral weighted shift operator with weight sequence (α_n); hence $T \in \mathfrak{L}(l^2)$. Let $k \in \mathbb{N}$ and define T_k as follows:

$$T_k e_n = \begin{cases} 0 & \text{if } n = 2^k(2l+1) \text{ for some } \ l \\ \alpha_n e_{n+1} & \text{otherwise} \end{cases}$$

Clearly, $T_k \in \mathfrak{L}(l^2)$ and

$$(T - T_k)e_n = \begin{cases} \alpha_n e_{n+1} & \text{if } n = 2^k(2l+1) \text{ for some } \ l \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} e^{-k} e_{n+1} & \text{if } n = 2^k(2l+1) \text{ for some } \ l \\ 0 & \text{otherwise.} \end{cases}$$

Let $(\xi_j) \in l^2$. Then, for each $k \in \mathbb{N}$,

$$\begin{aligned} ||(T - T_k)(\xi_j)||^2 = \left|\left|\sum_{j=1}^{\infty} \xi_j (T - T_k) e_j\right|\right|^2 &= \left|\left|\sum_{\substack{j=2^k(2l+1) \\ \text{for some } l}} \xi_j e^{-k} e_{j+1}\right|\right|^2 \\ &= \sum_{\substack{j=2^k(2l+1) \\ \text{for some } l}} |\xi_j e^{-k}|^2 \\ &\leq \sum_{j=1}^{\infty} |\xi_j|^2 (e^{-k})^2 \\ &= (e^{-k})^2 ||(\xi_j)||^2, \end{aligned}$$

so that $||T - T_k|| \leq e^{-k}$, and hence $T_k \to T$ as $k \to \infty$.

It can be verified that, if $n \in \mathbb{N}$, then $T_k^{2^k+1} e_n = 0$, for all $k \in \mathbb{N}$. Hence T_k is nilpotent and hence generalized Drazin invertible with $T_k^D = 0$.

We show that T is not generalized Drazin invertible. By Example 2.8.4, we have that $\sigma(T) = \overline{\mathcal{B}}(0, r(T))$. The result will follow from Theorem 5.1.16 if we can show that $r(T) \neq 0$. By the definition of T we have that

$$T^m e_n = \alpha_n \alpha_{n+1} \ldots \alpha_{n+m-1} e_{n+m}.$$

Let $(\xi_j) \in l^2$. Then

$$\begin{aligned}
||T^m(\xi_j)||^2 &= \left|\left|\sum_{j=1}^{\infty} \xi_j T^m e_j\right|\right|^2 \\
&= \left|\left|\sum_{j=1}^{\infty} \xi_j(\alpha_j\alpha_{j+1}\dots\alpha_{j+m-1}e_{j+m})\right|\right|^2 \\
&= \sum_{j=1}^{\infty} |\xi_j(\alpha_j\alpha_{j+1}\dots\alpha_{j+m-1})|^2 \\
&\leq (\sup\{\alpha_j\alpha_{j+1}\dots\alpha_{j+m-1} : j \in \mathbb{N}\})^2||(\xi_j)||^2,
\end{aligned}$$

so that $||T^m|| \leq \sup\{\alpha_j\alpha_{j+1}\dots\alpha_{j+m-1} : j \in \mathbb{N}\}$. Let $j \in \mathbb{N}$. Then

$$\frac{||T^m e_j||}{||e_j||} \leq \sup\left\{\frac{||T^m(\xi_j)||}{||(\xi_j)||} : 0 \neq (\xi_j) \in l^2\right\} = ||T^m||,$$

that is, $\alpha_j\alpha_{j+1}\dots\alpha_{j+m-1} \leq ||T^m||$ for all $j \in \mathbb{N}$, so that $\sup\{\alpha_j\alpha_{j+1}\dots\alpha_{j+m-1} : j \in \mathbb{N}\} \leq ||T^m||$, and hence

$$||T^m|| = \sup\{\alpha_j\alpha_{j+1}\dots\alpha_{j+m-1} : j \in \mathbb{N}\}.$$

Now, by using the definition of α_n, one can show that

$$\alpha_1\alpha_2\dots\alpha_{2^t-1} = \prod_{j=1}^{t-1} e^{-j2^{t-1-j}}.$$

By also using the facts that e^x is an increasing function on $\mathbb{R}$ and that $2^{t-1} < 2^t - 1$ for all $t \geq 2$, we have that

$$\begin{aligned}
(\alpha_1\alpha_2\dots\alpha_{2^t-1})^{\frac{1}{2^t-1}} &= \left(\prod_{j=1}^{t-1} e^{-j2^{t-1-j}}\right)^{\frac{1}{2^t-1}} \\
&= \prod_{j=1}^{t-1} e^{\frac{-j2^t}{2^{j+1}(2^t-1)}} \\
&= \prod_{j=1}^{t-1} e^{\frac{-j22^{t-1}}{2^{j+1}(2^t-1)}} \\
&> \prod_{j=1}^{t-1} e^{\frac{-j2}{2^{j+1}}} \\
&= \left(\prod_{j=1}^{t-1} e^{\frac{-j}{2^{j+1}}}\right)^2 \\
&= e^{-2\sum_{j=1}^{t-1}\frac{j}{2^{j+1}}},
\end{aligned}$$

and hence

$$e^{-2\beta} \leq \lim_{t\to\infty} (\alpha_1\alpha_2 \ldots \alpha_{2^t-1})^{\frac{1}{2^t-1}},$$

where $\beta = \sum_{j=1}^{\infty} \frac{j}{2^{j+1}}$. Let $m = 2^t - 1$. Then

$$\begin{aligned} 0 < e^{-2\beta} &\leq \lim_{m\to\infty} (\alpha_1\alpha_2 \ldots \alpha_m)^{\frac{1}{m}} \\ &\leq \lim_{m\to\infty} \sup\{\alpha_n\alpha_{n+1} \ldots \alpha_{n+m-1} : n \in \mathbb{N}\}^{\frac{1}{m}} \\ &= \lim_{m\to\infty} ||T^m||^{\frac{1}{m}} = r(T). \end{aligned}$$

It then follows that $r(T) \neq 0$.

Hence, we have a convergent sequence (T_k) of generalized Drazin invertible elements such that $\sup\{||T_k^D|| : k \in \mathbb{N}\} = 0$, but the limit T of (T_k) is not generalized Drazin invertible. ■

Finally, we investigate whether an analogous result of Lemma 2.7.3 is possible for group, Drazin and generalized Drazin inversion in Banach algebras.

Example 6.2.7 *Lemma 2.7.3 does not hold for group invertibility in general.*

Proof:
Consider the Banach algebra $C[0,1]$. Let $f_n = \frac{1}{n}f$, where $f \in C[0,1]$ is defined by $f(x) = x$. Then $\sigma(f_n) = [0, \frac{1}{n}]$. Moreover, (f_n) is a sequence in $C[0,1]$ which converges to 0. Although 0 is group invertible, we have by Lemma 3.2.1 that, for all $n \in \mathbb{N}$, f_n is not group invertible. ■

Remark 6.2.8 *First note that $C[0,1]$ is a semisimple commutative Banach algebra. Since group invertibility $\Leftrightarrow$ Drazin invertibility $\Leftrightarrow$ generalized Drazin invertibility in a semisimple commutative Banach algebra (see Corollary 5.1.18), we have from Example 6.2.7 that Lemma 2.7.3 holds neither for Drazin invertibility nor for generalized Drazin invertibility in general.*

The main reason why Lemma 2.7.3 fails to hold for group, Drazin and generalized Drazin invertibility is because the sets A^g, A^d and A^D (unlike A^{-1}), for a Banach algebra A need not be open in general, as illustrated in the following example.

Example 6.2.9 *([12], Example 8.4) Let A be a Banach algebra. The set A^g is not open in general.*

Consider the Banach algebra $C[0,1]$. By the definition of group inverses, 0 is group invertible with $0^g = 0$. We show that there exists no open ball B centred at 0 such that $B \subseteq C[0,1]^g$. Let $\epsilon > 0$. Then $\mathcal{B}(0,\epsilon)$ is an arbitary open ball around 0. Define $x \in C[0,1]$ by $x(t) = \frac{\epsilon}{2}t$. Then $|x(t)| \leq \frac{\epsilon}{2}$, so that $||x|| \leq \frac{\epsilon}{2} < \epsilon$; hence $x \in \mathcal{B}(0,\epsilon)$. Moreover, $\sigma(x) = [0, \frac{\epsilon}{2}]$; hence $0 \in \sigma(x) \backslash \text{iso}\ \sigma(x)$, so that x is not group invertible by Lemma 3.2.1. Since, for all $\epsilon > 0$, the open ball $\mathcal{B}(0,\epsilon) \nsubseteq C[0,1]^g$, we have that $C[0,1]^g$ is not open. ■

Remark 6.2.10 *By Corollary 5.1.18 and Remark 6.2.8, $C[0,1]^g = C[0,1]^d = C[0,1]^D$. Since $C[0,1]^g$ is not open (see content of Example 6.2.9), we conclude that the sets of Drazin and generalized Drazin invertible elements are not open in general.*

We give another example to show that an analogue of Lemma 2.7.3 is not available for generalized Drazin inverses.

Example 6.2.11 *([13], Example 2.1) Lemma 2.7.3 does not hold for generalized Drazin invertibility in general.*

Consider the Banach algebra $C\left[[0,1]\cup[2,3]\right]$. Define a_n and a by

$$a_n(t) = \begin{cases} \frac{1}{n}t & \text{if } t \in [0,1] \\ t & \text{if } t \in [2,3] \end{cases}$$

and

$$a(t) = \begin{cases} 0 & \text{if } t \in [0,1] \\ t & \text{if } t \in [2,3], \end{cases}$$

respectively. Then $a_n, a \in C\left[[0,1]\cup[2,3]\right]$ and $a_n \to a$ as $n \to \infty$. We also have that $\sigma(a_n) = a_n[[0,1]\cup[2,3]] = [0,\frac{1}{n}]\cup[2,3]$ and $\sigma(a) = \{0\}\cup[2,3]$. Since $0 \in \operatorname{iso}\sigma(a)$, we have from Theorem 5.1.16 that a is generalized Drazin invertible with

$$a^d(t) = \begin{cases} 0 & \text{if } t \in [0,1] \\ \frac{1}{t} & \text{if } t \in [2,3]. \end{cases}$$

However, by using Theorem 5.1.16 again, it is clear that none of the a_n's are generalized Drazin invertible. ■

In Examples 6.2.1 and 6.2.2 we saw that group, Drazin and generalized Drazin inversion in Banach algebras is not continuous in general. Thus, our concern is finding criteria for continuity of group, Drazin and generalized Drazin inversion. We will close this section with a continuity result, due to Koliha and Rakočević, for generalized Drazin invertibility in Banach algebras and mention that criteria for continuity of group and Drazin inversion for socle elements of a semisimple Banach algebra will be provided in the next section.

In Lemma 6.2.12, a result due to Roch and Silbermann (see [25]), we characterize the convergence of the group inverses of a convergent sequence of group invertible Banach algebra elements (a_n) in terms of the convergence of the group idempotents of a_n.

Lemma 6.2.12 *([25], Lemma 11) Let A be a Banach algebra and let (a_n) be a convergent sequence in A with limit a such that every a_n and a are group invertible. Then the group inverses of a_n converge to the group inverse of a if and only if the group idempotents of a_n converge to the group idempotent of a.*

Proof:
Suppose that $a_n \to a$ as $n \to \infty$, where a and all a_n are group invertible in A.

Suppose that $a_n^g \to a^g$ as $n \to \infty$. By using Lemma 3.1.6 and Corollaries 3.1.7 and 3.1.8, we have that $p_n = \mathbf{1} - a_n^g a_n$ and $p = \mathbf{1} - a^g a$ are the group idempotents of a_n and a respectively. It now follows that

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} (\mathbf{1} - a_n^g a_n) = \mathbf{1} - a^g a = p.$$

Conversely, suppose that p_n and p are the group idempotents of a_n and a respectively and that $p_n \to p$ as $n \to \infty$. Using Lemma 3.1.6 again, we have that $a_n^g = (a_n + p_n)^{-1}(\mathbf{1} - p_n)$ and $a^g = (a+p)^{-1}(\mathbf{1} - p)$ are the corresponding group inverses of a_n and a. The result then follows from the fact that $a_n \to a$, $p_n \to p$ and the continuity of inversion in Banach algebras. ■

Our main result in this section is Theorem 6.2.13. This result, due to Koliha and Rakočević, presents a number of continuity properties of the generalized Drazin inverse of a Banach algebra element. We will, in particular, see that an analogue of Lemma 6.2.12 is available for generalized Drazin inverses.

Recall that, in Example 6.2.6 we saw that even Lemma 2.7.2 is not possible for generalized Drazin inverses in general. Theorem 6.2.13, however, establishes an analogue of Lemma 2.7.2 for generalized Drazin inverses under the assumption that the limit of the convergent sequence of generalized Drazin invertible elements is also generalized Drazin invertible. Let us mention that the results obtained in Theorem 6.2.13 extend those of Rakočević in ([24], Theorem 4.1) for Drazin inverses.

Theorem 6.2.13 *([13], Theorem 2.4) Let A be a Banach algebra, $(a_n) \subseteq A$ a convergent sequence with limit $a \in A$, and suppose all a_n and a are generalized Drazin invertible with p_n and p the generalized Drazin idempotents of a_n and a respectively. Then the following statements are equivalent:*
(a) $a_n^D \to a^D$ *as* $n \to \infty$.
(b) $\sup\{||a_n^D|| : n \in \mathbb{N}\} < \infty$.
(c) $\sup\{r(a_n^D) : n \in \mathbb{N}\} < \infty$.
(d) $\inf\{\mathrm{D}(0, \sigma'(a_n)) : n \in \mathbb{N}\} > 0$.
(e) There exists an $r > 0$ *such that* $\mathcal{B}'(0, r) \subseteq \rho(a) \cap \left(\bigcap_{n=1}^{\infty} \rho(a_n)\right)$.
(f) $a_n^D a_n \to a^D a$ *as* $n \to \infty$.
(g) $p_n \to p$ *as* $n \to \infty$.

Proof:
Suppose that $a_n \to a$ as $n \to \infty$, where a and all a_n are generalized Drazin invertible in A. Denote by p and all p_n the generalized Drazin idempotents of a and a_n respectively.

The left-to-right implication (a)⇒(b) follows from the fact that convergence implies boundedness.

Suppose that (b) holds. Since $r(a_n^D) \leq ||a_n^D|| \leq \sup\{||a_n^D|| : n \in \mathbb{N}\}$ for all

$n \in \mathbb{N}$, we have that $\sup\{r(a_n^D) : n \in \mathbb{N}\} \leq \sup\{||a_n^D|| : n \in \mathbb{N}\} < \infty$, hence (b)$\Rightarrow$(c).

We show that (c)$\Rightarrow$(d): Suppose that $k = \sup\{r(a_n^D) : n \in \mathbb{N}\} < \infty$. We look at the following three cases:

(i) Suppose that $r(a_n^D) = 0$ for all $n \in \mathbb{N}$. Then $\sigma(a_n^D) = \{0\}$, and hence $\sigma(a_n) = \{0\}$, by Theorem 5.4.1, so that $\mathrm{D}(0, \sigma'(a_n)) = \infty$, for all $n \in \mathbb{N}$. It then follows that $\inf\{\mathrm{D}(0, \sigma'(a_n)) : n \in \mathbb{N}\} = \infty > 0$.

(ii) If $r(a_n^D) > 0$ for all $n \in \mathbb{N}$, then $k > 0$ and $\frac{1}{r(a_n^D)} \geq \frac{1}{k}$. Using Corollary 2.2.6, we have that $r(a_n^D) = r(a_n^D a_n a_n^D) \leq r(a_n^D)^2 r(a_n)$, so that $r(a_n) \geq \frac{1}{r(a_n^D)} > 0$ for all $n \in \mathbb{N}$. By Corollary 5.4.2, we have that $\mathrm{D}(0, \sigma'(a_n)) = \frac{1}{r(a_n^D)} \geq \frac{1}{k}$ for all $n \in \mathbb{N}$; hence

$$\inf\{\mathrm{D}(0, \sigma'(a_n)) : n \in \mathbb{N}\} \geq \frac{1}{k} > 0.$$

(iii) Suppose that there is at least one n such that $r(a_n^D) > 0$ and possibly some n for which $r(a_n^D) = 0$. By case (ii), we have that $\mathrm{D}(0, \sigma'(a_n)) = \frac{1}{r(a_n^D)} \geq \frac{1}{k}$, for all n satisfying $r(a_n^D) > 0$. By case (i), $\mathrm{D}(0, \sigma'(a_n)) = \infty$ for all n such that $r(a_n^D) = 0$. Let $k_n = \mathrm{D}(0, \sigma'(a_n))$. Then

$$\inf\{\mathrm{D}(0, \sigma'(a_n)) : n \in \mathbb{N}\} = \inf\{k_n : n \text{ satisfies } r(a_n^D) > 0\} \geq \frac{1}{k} > 0,$$

hence the result follows.

We show that (d)$\Rightarrow$(e): Suppose that $k = \inf\{\mathrm{D}(0, \sigma'(a_n)) : n \in \mathbb{N}\} > 0$. Let $r = \min\{s, k\}$ with $s = \mathrm{D}(0, \sigma'(a))$. By the choice of r and the fact that $\cap_{n=1}^{\infty} \rho(a_n) = \mathbb{C} \backslash \cup_{n=1}^{\infty} \sigma(a_n)$, we have that $\mathcal{B}'(0, r) \subseteq \bigcap_{n=1}^{\infty} \rho(a_n)$ and $\mathcal{B}'(0, r) \subseteq \rho(a)$, and hence $\mathcal{B}'(0, r) \subseteq \rho(a) \cap \left(\bigcap_{n=1}^{\infty} \rho(a_n)\right)$.

To prove the implication (e)$\Rightarrow$(f), suppose that there exists an $r > 0$ such that $\mathcal{B}'(0, r) \subseteq \rho(a) \cap \left(\bigcap_{n=1}^{\infty} \rho(a_n)\right)$. We are required to prove that $a_n^D a_n \to a^D a$ as $n \to \infty$.

If $a \in A^{-1}$, then, by Lemma 2.7.3, we have that $a_n \in A^{-1}$ for all sufficiently large n and $a_n^D = a_n^{-1} \to a^{-1} = a^D$ as $n \to \infty$. By the continuity of multiplication in a Banach algebra it follows that $a_n^D a_n \to a^D a$ as $n \to \infty$.

Suppose now that $0 \in \operatorname{iso} \sigma(a)$. By Corollary 5.1.8, p is the spectral idempotent of a corresponding to 0. Let $\Omega_0 = \{\lambda : |\lambda| < \frac{1}{3}r\}$, $\Omega_1 = \{\lambda : |\lambda| > \frac{2}{3}r\}$ and $\Omega = \Omega_0 \cup \Omega_1$. By hypothesis, we have that Ω_0 and Ω_1 are open sets containing $\{0\}$ and $\sigma'(a)$, respectively, and hence Ω is an open set containing $\sigma(a)$. Define $f : \Omega \to \mathbb{C}$ by

$$f(\lambda) = \begin{cases} 0 & \text{if } \lambda \in \Omega_1 \\ 1 & \text{if } \lambda \in \Omega_0. \end{cases}$$

Then $f \in H(\Omega)$ and $p = f(a)$ by Theorem 2.3.3. From Corollary 5.1.8 we have that p_n is the spectral idempotent of a_n corresponding to 0; where it should be noted that p_n might be 0 for several n. Also, by hypothesis, Ω_1 is an open set containing $\sigma'(a_n)$ for all $n \in \mathbb{N}$, so that $p_n = f(a_n)$ for all $n \in \mathbb{N}$. It then follows from HFC(6) that $p_n = f(a_n) \to f(a) = p$ as $n \to \infty$, and hence $a_n^D a_n \to a^D a$ as $n \to \infty$ by Corollary 5.1.8. (We remark that as a result of Newburgh's theorem (Theorem 2.4.3), there cannot be infinitely many n for which a_n is invertible).

By Corollary 5.1.8, $p_n = \mathbf{1} - a_n^D a_n$ and $p = \mathbf{1} - a^D a$. Hence (f)$\Leftrightarrow$(g).

Finally, to prove that (g)$\Rightarrow$(a), suppose that (g) holds. Then $a_n+p_n \to a+p$ as $n \to \infty$, where $a+p \in A^{-1}$ by Proposition 5.1.4. From Lemma 2.7.3 it then follows that $a_n+p_n \in A^{-1}$ for all sufficiently large n and $(a_n+p_n)^{-1} \to (a+p)^{-1}$ as $n \to \infty$. Moreover, $a_n^D = (a_n + p_n)^{-1}(\mathbf{1} - p_n)$ and $a^D = (a + p)^{-1}(\mathbf{1} - p)$ by Corollary 5.1.8. Hence, by the continuity of multiplication in a Banach algebra, (a) holds. This completes the proof. ■

Since $A^g \subseteq A^d \subseteq A^D$, for an arbitrary Banach algebra A, Theorem 6.2.13 presents various continuity properties for both group and Drazin inversion in Banach algebras. Let us mention that several of the continuity properties obtained in Theorem 6.2.13 will be useful in the next section.

Remark 6.2.14 *Observe that the equivalence of (a) and (g) in Theorem 6.2.13 is an analogue of Lemma 6.2.12 for generalized Drazin inverses. Note also that a similar result to Proposition 6.2.3 and Corollary 6.2.4 (see also Lemma 2.7.2) is obtained for generalized Drazin inverses (the equivalence of (a) and (b) in Theorem 6.2.13) when assuming that the limit of the convergent sequence of generalized Drazin invertible elements is also generalized Drazin invertible.*

6.3 A generalization of Campbell and Meyer's result due to the spectral rank

In this section our aim is to present criteria, using the concept of the spectral rank, for continuity of group and Drazin inversion in the special case of the socle elements in semisimple Banach algebras, as was done by the authors in [7]. Our main results in this section include Theorem 6.3.5 and Corollary 6.3.6. Theorem 6.3.5 is a special case of Corollary 6.3.6, where Corollary 6.3.6, as we will see, is a generalization of Campbell and Meyer's continuity result for square matrices (see Theorem 2.12.4) to arbitrary elements of the socle of a semisimple Banach algebra.

The following result was proved by Brits, Lindeboom and Raubenheimer in [6]. We will formulate and prove only the one implication of the result, which shows that the socle elements of a semisimple Banach algebra are a special class of Drazin invertible elements.

Theorem 6.3.1 *([6], Theorem 9) Let A be a semisimple Banach algebra. Then* $\mathrm{Soc}(A) \subseteq A^d$.

Proof:
Suppose that $a \in \mathrm{Soc}(A)$. From Theorem 5.1.16 and Corollary 2.5.6 we have

that $a \in A^D$, and hence $a - aa^Da \in \mathrm{QN}(A)$. Since $a \in \mathrm{Soc}(A)$ and $\mathrm{Soc}(A)$ is a two-sided ideal, we have that $a - aa^Da \in \mathrm{Soc}(A)$, so that $a - aa^Da \in \mathrm{Soc}(A) \cap \mathrm{QN}(A)$. From Corollary 2.5.5 it then follows that $a - aa^Da \in \mathrm{N}(A)$. Recalling the definition of the Drazin inverse we have that that $a \in A^d$. Hence $\mathrm{Soc}(A) \subseteq A^d$. ∎

The following result will be required in the proof of Theorem 6.3.5. Recall Lemma 3.1.3(2) which implies that, for $a \in A^g$, the element aa^g is an idempotent of an algebra A.

Theorem 6.3.2 *([7], Theorem 2.4) Let A be a semisimple Banach algebra and $a \in \mathrm{Soc}(A) \cap A^g$. Then $\mathrm{rank}(a) = \mathrm{rank}(a^k)$ for all $k \in \mathbb{N}$.*

Proof:
Suppose that $a \in \mathrm{Soc}(A) \cap A^g$. By Lemma 3.1.3(1), a^k is group invertible with group inverse $(a^g)^k$. Due to the fact that we are dealing with more than one semisimple Banach algebra in this proof, we will, to avoid any confusion, from this point on write $\mathrm{rank}_A(a)$ to mean $\mathrm{rank}(a)$. Let $p = aa^g$. Then pAp is a semisimple Banach algebra with identity p. Observe that $a = pap$ is invertible in pAp with inverse $a^g = pa^gp$. It then follows that a^k is invertible in pAp with inverse $(a^g)^k$. Moreover, by Proposition 2.6.2(c) and Lemma 2.6.4, we have that

$$\mathrm{rank}_A(a) = \mathrm{rank}_{pAp}(a) = \mathrm{rank}_{pAp}(p) = \mathrm{rank}_{pAp}(a^k) = \mathrm{rank}_A(a^k).$$

This completes the proof. ∎

Observe that, for a semisimple Banach algebra A, since every idempotent is group invertible, we have from Theorem 2.6.8 and Lemma 3.1.10 that every maximal finite-rank element is group invertible in $\mathrm{Soc}(A)$; hence $\mathrm{rank}(a^{(c)}) = \mathrm{rank}(a)$ for all maximal finite-rank elements a. In our next result we show that the equality $\mathrm{rank}(a^{(c)}) = \mathrm{rank}(a)$, where a is a maximal finite-rank element, can also be obtained without using Theorem 2.6.8.

Lemma 6.3.3 *Let A be a semisimple Banach algebra and $a \in A$. If a is a maximal finite-rank element, then $a^{(c)}$ is a maximal finite-rank element. Moreover, $\mathrm{rank}(a^{(c)}) = \mathrm{rank}(a)$.*

Proof:
Suppose that $a \in A$ is a maximal finite-rank element. From Theorem 6.3.1 it follows that $a \in A^d$. By the definition of the spectral rank we have that $\#\sigma'(a^{(c)}) \leq \mathrm{rank}(a^{(c)})$. Hence, by hypothesis, Proposition 2.6.2(a), Corollaries 5.3.2 and 5.3.4 and the remark preceding Corollary 5.3.4, we have that

$$\mathrm{rank}(a^{(c)}) = \mathrm{rank}(aa^da) \leq \mathrm{rank}(a) = \#\sigma'(a) = \#\sigma'(a^{(c)}) \leq \mathrm{rank}(a^{(c)}).$$

Hence $a^{(c)}$ is maximal finite-rank and $\mathrm{rank}(a^{(c)}) = \mathrm{rank}(a)$. ∎

In order to prove Corollary 6.3.6, we need Theorem 6.3.5, which relies on Lemma 6.3.4. Let us remark that Theorem 6.3.5 is a special case of Corollary 6.3.6. Under the assumption that $a_n \to a$ in $\mathrm{Soc}(A)$ as $n \to \infty$, our aim is to provide conditions which will ensure that $a_n^d \to a^d$ as $n \to \infty$. In Lemma 6.3.4 we restrict ourselves to maximal finite-rank elements, which, as remarked, form a special class of group invertible elements. In [7] Brits, Lindeboom and Raubenheimer only showed the one implication ((b)⇒(a)) in Lemma 6.3.4. We observe that the reverse implication also holds.

Lemma 6.3.4 *([7], Lemma 3.1) Let A be a semisimple Banach algebra and (a_n) a convergent sequence in* $\mathrm{Soc}(A)$ *with limit $a \in \mathrm{Soc}(A)$, where a and all a_n are maximal finite-rank elements. The following statements are equivalent:*
(a) $a_n^g \to a^g$ as $n \to \infty$.
(b) There exists $n_0 \in \mathbb{N}$ such that $\mathrm{rank}(a_n) = \mathrm{rank}(a)$ for all $n \geq n_0$.

Proof:
Suppose that a_n and a are maximal finite-rank elements and that $a_n \to a$ in $\mathrm{Soc}(A)$ as $n \to \infty$.

(a)⇒(b): Suppose that $a_n^g \to a^g$ as $n \to \infty$. By the continuity of multiplication in a Banach algebra, we have that $a_n a_n^g \to aa^g$ as $n \to \infty$. Hence we can find an $n_0 \in \mathbb{N}$ such that $||a_n a_n^g - aa^g|| < \frac{1}{||2(aa^g)-1||}$, for all $n \geq n_0$. By Lemma 3.1.3(2), $a_n a_n^g$ and aa^g are idempotents and hence from Lemma 2.1.15 it follows that $a_n a_n^g$ and aa^g are similar for all $n \geq n_0$. Let $x_n \in A^{-1}$ be such that $aa^g = x_n^{-1} a_n a_n^g x_n$ for all $n \geq n_0$. It follows from Proposition 2.6.2(b) that

$$\mathrm{rank}(aa^g) = \mathrm{rank}(x_n^{-1} a_n a_n^g x_n) = \mathrm{rank}(a_n a_n^g x_n) = \mathrm{rank}(a_n a_n^g),$$

for all $n \geq n_0$ and hence by Proposition 2.6.2(a) we have that

$$\mathrm{rank}(a_n) = \mathrm{rank}(a_n a_n^g a_n) \leq \mathrm{rank}(a_n a_n^g) = \mathrm{rank}(aa^g) \leq \mathrm{rank}(a) = \mathrm{rank}(aa^g a)$$
$$\leq \mathrm{rank}(aa^g) = \mathrm{rank}(a_n a_n^g) \leq \mathrm{rank}(a_n),$$

for all $n \geq n_0$. Hence $\mathrm{rank}(a_n) = \mathrm{rank}(a)$ for all $n \geq n_0$.

(b)⇒(a): Suppose that there exists $n_0 \in \mathbb{N}$ such that $\mathrm{rank}(a_n) = \mathrm{rank}(a)$ for all $n \geq n_0$. Since a and all a_n are maximal finite-rank elements, we have that $\mathrm{rank}(a_n) = \#\sigma'(a_n)$ and $\mathrm{rank}(a) = \#\sigma'(a)$, respectively. Hence, for all $n \geq n_0$, the equality $\#\sigma'(a_n) = \#\sigma'(a)$ holds. By Corollary 2.4.4, $\inf\{\mathrm{D}(0, \sigma'(a_n)) : n \in \mathbb{N}\} > 0$, and hence from Theorem 6.2.13 it follows that $a_n^g \to a^g$ as $n \to \infty$.
■

We are now ready to present our first main result in this section. This result, by Brits, Lindeboom and Raubenheimer, extends Lemma 6.3.4 to arbitrary group invertible elements in $\mathrm{Soc}(A)$ and presents criteria for continuity of group inversion of group invertible socle elements in semisimple Banach algebras.

Theorem 6.3.5 *([7], Theorem 3.2) Let A be a semisimple Banach algebra and (a_n) a convergent sequence in* $\mathrm{Soc}(A) \cap A^g$ *with limit* $a \in \mathrm{Soc}(A) \cap A^g$*. Then the following statements are equivalent:*
(a) $a_n^g \to a^g$ *as* $n \to \infty$.
(b) There exists $n_0 \in \mathbb{N}$ *such that* $\mathrm{rank}(a_n) = \mathrm{rank}(a)$ *for all* $n \geq n_0$.

Proof:
Suppose that $a_n \to a$ in $\mathrm{Soc}(A) \cap A^g$ as $n \to \infty$.

The proof of the implication (a)⇒(b) was done in Lemma 6.3.4[(a)⇒(b)].

(b)⇒(a): Suppose that there exists $n_0 \in \mathbb{N}$ such that $\mathrm{rank}(a_n) = \mathrm{rank}(a)$ for all $n \geq n_0$. Due to the fact that we are dealing with more than one semisimple Banach algebra in this proof, we will, to avoid any confusion, from this point on write $\mathrm{rank}_A(a)$ to mean $\mathrm{rank}(a)$. Let $q_n = a_n a_n^g$ and $q = aa^g$. Then $B_n = q_n A q_n$ and $B = qAq$ are semisimple Banach algebras with identities q_n and q respectively. Since $a_n, a \in \mathrm{Soc}(A)$, also $q_n, q \in \mathrm{Soc}(A)$, and hence it follows from Proposition 2.5.3 that $\dim B_n < \infty$ and $\dim B < \infty$. By the Wedderburn-Artin theorem (Theorem 2.2.8) B_n and B are direct sums of matrix algebras over $\mathbb{C}$.

Since $qaq = aa^g aaa^g = aa^g a = a$ and, similarly, $q_n a_n q_n = a_n$, we have that $a \in B$ and $a_n \in B_n$. One can also easily verify that a_n and a are invertible in B_n and B with inverses $a_n^g = q_n a_n^g q_n$ and $a^g = qa^g q$ respectively.

By Theorem 2.6.7 we have that $E(a), E(a_n)(n \in \mathbb{N}), E(a^2)$, $E(a_n^2)(n \in \mathbb{N}), E(q)$ and $E(q_n)(n \in \mathbb{N})$ are dense open subsets of A. By Baire's Category Theorem (Theorem 2.1.6) it follows that

$$\overline{\cap_{n=1}^{\infty}[E(a) \cap E(a_n) \cap E(a^2) \cap E(a_n^2) \cap E(q) \cap E(q_n)]} = A,$$

and hence the intersection in the expression on the left hand side is non-empty. Let x be an element of this set.

Using the remark following the definition of the spectrum and the fact that $x \in E(a)$, we have that

$$\begin{aligned} \mathrm{rank}_A(a) &= \#\sigma'_A(ax) \\ &= \#\sigma'_A(qaqx) \\ &= \#\sigma'_A(aqxq). \end{aligned} \tag{6.3.1}$$

Similarly, because $x \in E(a_n)$, we have that

$$\mathrm{rank}_A(a_n) = \#\sigma'_A(a_n q_n x q_n), \tag{6.3.2}$$

for all $n \in \mathbb{N}$. Now, recalling (6.3.1) and Lemmas 2.2.9 and 2.6.4, it follows that

$$\mathrm{rank}_A(a) = \#\sigma'_A(aqxq) = \#\sigma'_B(aqxq) = \mathrm{rank}_B(aqxq) = \mathrm{rank}_A(aqxq)$$

and, by also using Proposition 2.6.2(a) and Lemma 2.6.4 again,

$$\operatorname{rank}_A(aqxq) = \operatorname{rank}_B(aqxq) \leq \operatorname{rank}_B(a) = \operatorname{rank}_A(a),$$

so that

$$\operatorname{rank}_A(aqxq) \quad = \quad \operatorname{rank}_A(a). \tag{6.3.3}$$

A similar argument, using equation (6.3.2), shows that

$$\operatorname{rank}_A(a_nq_nxq_n) \quad = \quad \operatorname{rank}_A(a_n) \tag{6.3.4}$$

for all $n \in \mathbb{N}$. By Lemmas 2.2.9 and 2.6.4 and equations (6.3.1), (6.3.2), (6.3.3) and (6.3.4) it is clear that $aqxq$ and all $a_nq_nxq_n$ are maximal finite-rank elements in B and B_n, respectively.

From equation (6.3.1) and Lemmas 2.2.9 and 2.6.4 it follows that $\operatorname{rank}_B(a) = \#\sigma'_B(a(qxq))$. Hence, by the Wedderburn-Artin theorem, this, together with the facts that $a \in B^{-1}$ and $aqxq$ is a maximal finite-rank element in B, implies that $qxq \in B^{-1}$, if we apply Lemma 2.6.10. Analogously, using (6.3.2), we have that $q_nxq_n \in B_n^{-1}$ for all $n \in \mathbb{N}$.

In order to prove the convergence of a_n^g we show that $a_na_n^g$ converge. To obtain the latter property we will first show that $(q_nxq_n)^{-1} \to (qxq)^{-1}$ as $n \to \infty$ and then conclude that $q_nxq_n \to qxq$ as $n \to \infty$.

Since $a_n \to a$ as $n \to \infty$, we have by the continuity of multiplication in a Banach algebra that

$$a_n(q_nxq_n)a_n = a_nxa_n \to axa = a(qxq)a$$

in $\operatorname{Soc}(A)$ as $n \to \infty$. By hypothesis, Proposition 2.6.2(b), Lemma 2.6.4 and the facts that $a_n, q_nxq_n \in B_n^{-1}$ and $a, qxq \in B^{-1}$, we have that

$$\begin{aligned}
\operatorname{rank}_A(a_nq_nxq_na_n) &= \operatorname{rank}_{B_n}(a_nq_nxq_na_n) \\
&= \operatorname{rank}_{B_n}(a_n) \\
&= \operatorname{rank}_A(a_n) \\
&= \operatorname{rank}_A(a) \\
&= \operatorname{rank}_B(a) \\
&= \operatorname{rank}_B(aqxqa) \\
&= \operatorname{rank}_A(aqxqa),
\end{aligned}$$

for all $n \geq n_0$. Recalling the remark following the definition of the spectrum and the fact that $x \in E(a^2)$, we have that

$$\begin{aligned}
\operatorname{rank}_A(a^2) &= \#\sigma'_A(a^2x) \\
&= \#\sigma'_A(qa^2qx) \\
&= \#\sigma'_A(a^2qxq) \\
&= \#\sigma'_A(aqxqa).
\end{aligned} \tag{6.3.5}$$

Similarly, using the fact that $x \in E(a_n^2)$, we obtain that

$$\operatorname{rank}_A(a_n^2) \;\; = \#\sigma'_A(a_n q_n x q_n a_n), \tag{6.3.6}$$

for all $n \in \mathbb{N}$. Now, using (6.3.5), Proposition 2.6.2(b) Lemma 2.6.4, Theorem 6.3.2 and the fact that $a, qxq \in B^{-1}$, it follows that

$$\begin{aligned}\operatorname{rank}_A(aqxqa) &= \operatorname{rank}_B(aqxqa)\\ &= \operatorname{rank}_B(a)\\ &= \operatorname{rank}_A(a)\\ &= \operatorname{rank}_A(a^2)\\ &= \#\sigma'_A(aqxqa).\end{aligned}$$

Similarly, by recalling equation (6.3.6) and the fact that $a_n, q_n x q_n \in B_n^{-1}$, we obtain that $\operatorname{rank}_A(a_n q_n x q_n a_n) = \#\sigma'_A(a_n q_n x q_n a_n)$ for all $n \in \mathbb{N}$, and hence $aqxqa$ and all $a_n q_n x q_n a_n$ are maximal finite-rank elements.

We have thus shown that the elements $a_n q_n x q_n a_n, aqxqa \in \operatorname{Soc}(A)$ satisfy the conditions in Lemma 6.3.4, and hence we conclude that $(a_n q_n x q_n a_n)^d \to (aqxqa)^d$ as $n \to \infty$.

Since $a_n(q_n x q_n)a_n$ and $a(qxq)a$ are invertible in B_n and B, respectively, it follows that $(aqxqa)^d = (aqxqa)^{-1} = a^{-1}(qxq)^{-1}a^{-1} = a^g(qxq)^{-1}a^g$, where the understanding is that, where the usual inverse notation is used, the element is invertible only in B_n or B. Similarly, $(a_n q_n x q_n a_n)^d = a_n^g(q_n x q_n)^{-1}a_n^g$ for all $n \in \mathbb{N}$. Hence, we have that

$$a_n^g(q_n x q_n)^{-1}a_n^g \to a^g(qxq)^{-1}a^g$$

as $n \to \infty$, so that, by the continuity of multiplication in a Banach algebra and the fact that $a_n a_n^g$ and aa^g are the identity elements in B_n and B respectively,

$$(q_n x q_n)^{-1} = a_n a_n^g (q_n x q_n)^{-1} a_n^g a_n \to aa^g(qxq)^{-1}a^g a = (qxq)^{-1} \tag{6.3.7}$$

as $n \to \infty$.

Note that $(qxq)^{-1} = (qxq)^{-1}q \in \operatorname{Soc}(A)$, and similarly $(q_n x q_n)^{-1} \in \operatorname{Soc}(A)$ for all n. By hypothesis, Proposition 2.6.2[(b) and (c)], Lemma 2.6.4 and the facts that $a_n^g, (q_n x q_n)^{-1} \in B_n^{-1}$ and $a^g, (qxq)^{-1} \in B^{-1}$, we have that

$$\operatorname{rank}_A(q_n x q_n)^{-1} = \operatorname{rank}_{B_n}(q_n x q_n)^{-1} = \operatorname{rank}_{B_n}(a_n a_n^g) = \operatorname{rank}_{B_n}(a_n) = \operatorname{rank}_A(a_n)$$

$$= \operatorname{rank}_A(a) = \operatorname{rank}_B(a) = \operatorname{rank}_B(aa^g) = \operatorname{rank}_B(qxq)^{-1} = \operatorname{rank}_A(qxq)^{-1},$$

for all $n \geq n_0$. Recalling the remark following the definition of the spectrum, Proposition 2.6.2(c), Lemma 2.6.4 and the facts that q is an idempotent and the identity of B and $x \in E(q)$, we have that

$$\begin{aligned} \operatorname{rank}_A(qxq)^{-1} &= \operatorname{rank}_B(qxq)^{-1} \\ &= \operatorname{rank}_B(q) \\ &= \operatorname{rank}_A(q) \\ &= \#\sigma'_A(qx) \\ &= \#\sigma'_A(qxq) \\ &= \#\sigma'_A(qxq)^{-1}, \end{aligned}$$

where the last step follows from the spectral mapping theorem (also see the proof of Corollary 5.4.2) and Lemma 2.2.9. Similarly, using the fact that $x \in E(q_n)$, where q_n is an idempotent and the identity of B_n, we obtain that $\operatorname{rank}_A(q_nxq_n)^{-1} = \#\sigma'_A(q_nxq_n)^{-1}$ for all $n \in \mathbb{N}$, and hence it follows that $(qxq)^{-1}$ and all $(q_nxq_n)^{-1}$ are maximal finite-rank elements.

Since the elements $(q_nxq_n)^{-1}$ and $(qxq)^{-1}$ satisfy the conditions in Lemma 6.3.4, we have that

$$q_nxq_n = ((q_nxq_n)^{-1})^{-1} = ((q_nxq_n)^{-1})^d \rightarrow ((qxq)^{-1})^d = ((qxq)^{-1})^{-1} = qxq \tag{6.3.8}$$

as $n \rightarrow \infty$.

Now, by the continuity of multiplication in a Banach algebra and equations (6.3.7) and (6.3.8), we have that

$$a_na_n^g = (q_nxq_n)(q_nxq_n)^{-1} \rightarrow (qxq)(qxq)^{-1} = aa^g$$

as $n \rightarrow \infty$, and hence $a_n^g \rightarrow a^g$ as $n \rightarrow \infty$, by Theorem 6.2.13. ■

In our next result we characterize the continuity of Drazin inversion of socle elements in semisimple Banach algebras in terms of the spectral rank.

Corollary 6.3.6 *([7], Corollary 3.3) Let A be a semisimple Banach algebra and (a_n) a convergent sequence in $\operatorname{Soc}(A)$ with limit $a \in \operatorname{Soc}(A)$, and suppose all a_n are Drazin invertible of uniformly bounded Drazin degree. Then the following statements are equivalent:*
(a) $a_n^d \rightarrow a^d$ as $n \rightarrow \infty$.
(b) There exists $n_0 \in \mathbb{N}$ such that $\operatorname{rank}\left(a_n^{(c)}\right) = \operatorname{rank}\left(a^{(c)}\right)$ for all $n \geq n_0$.

Proof:
Suppose that $a_n \rightarrow a$ in $\operatorname{Soc}(A)$ as $n \rightarrow \infty$ and that a and all a_n are Drazin invertible of uniformly bounded Drazin degree.

(a)⇒(b): Suppose that $a_n^d \rightarrow a^d$ as $n \rightarrow \infty$. By Corollaries 5.3.2 and 5.3.4, $a_n^{(c)} = a_na_n^da_n$ and $a^{(c)} = aa^da$. Moreover, $a_n^{(c)}$ and $a^{(c)}$ are group invertible elements belonging to the socle of A and by the continuity of multiplication in a Banach algebra we have that $a_n^{(c)} \rightarrow a^{(c)}$ as $n \rightarrow \infty$. Again by Corollary 5.3.4, we have that $\left(a_n^{(c)}\right)^g = a_n^d$ and $\left(a^{(c)}\right)^g = a^d$. By hypothesis $\left(a_n^{(c)}\right)^g \rightarrow$

$\left(a^{(c)}\right)^g$ as $n \to \infty$; hence by Theorem 6.3.5, there exists $n_0 \in \mathbb{N}$ such that $\operatorname{rank}\left(a_n^{(c)}\right) = \operatorname{rank}\left(a^{(c)}\right)$, for all $n \geq n_0$.

(b)$\Rightarrow$(a): Suppose that there exists $n_0 \in \mathbb{N}$ such that $\operatorname{rank}\left(a_n^{(c)}\right) = \operatorname{rank}\left(a^{(c)}\right)$ for all $n \geq n_0$. Let $l = \sup\{k_n : n \in \mathbb{N}\}$, where k_n is the smallest integer for which a_n is Drazin invertible and suppose that a is Drazin invertible of degree k. Let $m = \max\{k, l\}$. By Lemma 4.1.4, a and all a_n are Drazin invertible of degree m, and hence by hypothesis and Corollary 6.1.3 we have that a^m and all a_n^m are group invertible in $\operatorname{Soc}(A)$. From Corollary 4.1.7 it follows that $(a_n^m)^g = (a_n^m)^d = (a_n^d)^m$ and $(a^m)^g = (a^m)^d = (a^d)^m$.

Since a^m is group invertible, we have from Lemma 4.1.6 and Corollaries 5.3.2 and 5.3.4 that

$$a^{(c)} = aa^d a = a[(a^d)^m a^{m-1}]a = a(a^m)^g a^m.$$

By also using Proposition 2.6.2(a) it follows that

$$\operatorname{rank}\left(a^{(c)}\right) \leq \operatorname{rank}(a^m).$$

Since $a^{(c)}a^{(n)} = a^{(n)}a^{(c)} = 0$, we have from Corollary 5.3.4 and the binomial theorem that

$$\begin{aligned} a^m &= \left(a^{(c)} + a^{(n)}\right)^m \\ &= \sum_{k=0}^{m} \binom{m}{k} \left(a^{(c)}\right)^{m-k} \left(a^{(n)}\right)^k \\ &= \left(a^{(c)}\right)^m + \left(a^{(n)}\right)^m. \end{aligned}$$

Let p be the Drazin idempotent of a. By Corollary 5.1.7, p is the spectral idempotent of a corresponding to 0. Since a is Drazin invertible of degree m, we have from Proposition 4.1.9 and Corollaries 5.3.2 and 5.3.4 that $\left(a^{(n)}\right)^m = (ap)^m = 0$, and hence $a^m = \left(a^{(c)}\right)^m = a^{(c)}\left(a^{(c)}\right)^{m-1}$, so that

$$\operatorname{rank}(a^m) \leq \operatorname{rank}\left(a^{(c)}\right).$$

Hence, $\operatorname{rank}(a^m) = \operatorname{rank}\left(a^{(c)}\right)$. Analogously, $\operatorname{rank}(a_n^m) = \operatorname{rank}\left(a_n^{(c)}\right)$, so that

$$\operatorname{rank}(a_n^m) = \operatorname{rank}\left(a_n^{(c)}\right) = \operatorname{rank}\left(a^{(c)}\right) = \operatorname{rank}(a^m),$$

for all $n \geq n_0$.

By Theorem 6.3.5, $(a_n^m)^g \to (a^m)^g$ as $n \to \infty$ and by using the facts that multiplication in a Banach algebra is continuous and $a^d \in \operatorname{Comm} a$, we have that

$$(a_n a_n^d)^m = a_n^m (a_n^d)^m = a_n^m (a_n^m)^g \to a^m (a^m)^g = a^m (a^d)^m = (aa^d)^m$$

as $n \to \infty$. By Lemma 3.1.3(2), $a_n a_n^d$ and aa^d are idempotents, hence $(a_n a_n^d)^m = a_n a_n^d$ and $(aa^d)^m = aa^d$, so that $a_n a_n^d \to aa^d$ as $n \to \infty$. From Theorem 6.2.13 it follows that $a_n^d \to a^d$ as $n \to \infty$. This completes the proof. ■

Note that, for the semisimple Banach algebra $M_n(\mathbb{C})$, we have that the elements of any sequence in $M_n(\mathbb{C}) = \mathrm{Soc}(M_n(\mathbb{C}))$ are Drazin invertible of uniformly bounded Drazin degree. Hence Corollary 6.3.6, due to Brits, Lindeboom and Raubenheimer, generalizes Campbell and Meyer's result for square matrices (see Theorem 2.12.4) to arbitrary socle elements of a semisimple Banach algebra.

We remarked at the beginning of this section that Theorem 6.3.5 is a special case of Corollary 6.3.6. This can be seen from the facts that all $a_n \in A^g \cap \mathrm{Soc}(A)$ are Drazin invertible of uniformly bounded Drazin degree and that, for $a_n, a \in A^g$, $a = a^{(c)}$ and $a_n = a_n^{(c)}$ by Corollary 5.3.4.

Remark 6.3.7 *Let us remark that an operator version of Campbell and Meyer's result was given by Koliha and Rakočević in ([13], Theorem 5.1).*

Chapter 7

The generalized Drazin inverse for bounded linear operators

One of the most important Banach algebras is the Banach algebra $\mathfrak{L}(X)$ of all bounded linear operators on a complex Banach space X. In this chapter our aim is to present several continuity properties of the generalized Drazin inverse of a bounded linear operator. This chapter consists of three sections. The aim of Section 7.1 is to present some properties a generalized Drazin invertible operator in $\mathfrak{L}(X)$ have. In Section 7.2 we formulate and prove useful results results about the gap (see Definition 2.10.1), as was done in [13] by Koliha and Rakočević, that will be required in Section 7.3. Finally, in Section 7.3 we study the continuity properties for the generalized Drazin inverse of a bounded linear operator. The main result in this chapter is Theorem 7.3.3.

7.1 Generalized Drazin invertible operators in $\mathfrak{L}(X)$

In the 1977 paper of King (see [11]), he presented some of the properties Drazin invertible bounded linear operators have. This section is aimed at specializing some of the results obtained in Chapter 5 for arbitrary generalized Drazin invertible Banach algebra elements to the bounded linear operator case. We start by formulating Corollary 5.3.3 for the Banach algebra $\mathfrak{L}(X)$ and develop some results from it.

Theorem 7.1.1 *([12], Corollary 6.5) Let $A \in \mathfrak{L}(X)$. An element A is generalized Drazin invertible if and only if it is the sum of unique elements $C \in \mathfrak{L}(X)^g$ and $Q \in \mathrm{QN}(\mathfrak{L}(X))$ such that $CQ = QC = 0$. Then $A^D = C^g$.*

We remind the reader that the notations $A^{(c)}$ and $A^{(q)}$ are used to denote the core operator and the quasinilpotent part of $A \in \mathfrak{L}(X)$, respectively. Let us also mention that Corollary 7.1.2 was already observed in Chapter 5 for an arbitrary generalized Drazin invertible element in a Banach algebra. We present a different proof here using Corollary 2.2.6.

Corollary 7.1.2 *([13], p.173) If $A \in \mathfrak{L}(X)^D$, then $\sigma(A) = \sigma(A^{(c)})$.*

Proof:
Suppose that $A \in \mathfrak{L}(X)^D$. Since $A^{(q)}A^{(c)} = A^{(c)}A^{(q)}$, one can show that $A^{(q)}$ and A commute. Moreover, by Theorem 7.1.1, Corollary 2.2.6 and the fact that $A^{(q)} \in \mathrm{QN}(\mathfrak{L}(X))$, we have that $\sigma(A) = \sigma(A^{(c)} + A^{(q)}) \subseteq \sigma(A^{(c)}) + \sigma(A^{(q)}) = \sigma(A^{(c)})$. In a similar way, using the fact that $A^{(q)}$ and A commute, one can show that $\sigma(A^{(c)}) \subseteq \sigma(A)$. Hence $\sigma(A^{(c)}) = \sigma(A)$. ■

Corollary 7.1.3 *([13], p.174) Let $A \in \mathfrak{L}(X)^D$. If $0 \in \operatorname{iso}\sigma(A)$, then the spectral idempotent of A corresponding to 0 is also the spectral idempotent of $A^{(c)}$ corresponding to 0.*

Proof:
Suppose that $0 \in \operatorname{iso}\sigma(A)$ and let P_A denote the spectral idempotent of A corresponding to 0. By Corollary 5.1.8, $P_A = I - A^D A$. By Corollary 7.1.2, we have that $0 \in \operatorname{iso}\sigma(A^{(c)})$. Let $P_{A^{(c)}}$ denote the spectral idempotent of $A^{(c)}$ corresponding to 0. Again by Corollary 5.1.8, $P_{A^{(c)}} = I - {A^{(c)}}^D A^{(c)} = I - {A^{(c)}}^g A^{(c)}$. Now,

$$P_A = I - A^D A = I - {A^{(c)}}^g(A^{(c)} + A^{(q)}) = I - {A^{(c)}}^g A^{(c)} = P_{A^{(c)}}.$$

Hence the result follows. ■

Corollary 7.1.4 *([13], (4.1)) Let $A \in \mathfrak{L}(X)^D$. Then $\mathrm{Null}(A^{(c)}) = \mathrm{R}(P)$, where P denotes the generalized Drazin idempotent of A.*

Proof:
Suppose that $A \in \mathfrak{L}(X)^D$. If $A \in \mathfrak{L}(X)^{-1}$, then $A = A^{(c)}$ and $P = 0$ by Corollary 5.1.8, so that the result clearly holds in this case.

Suppose now that $0 \in \operatorname{iso}\sigma(A)$ and let P denote the spectral idempotent of A corresponding to 0. By Corollary 7.1.3, P is the spectral idempotent of $A^{(c)}$ corresponding to 0, and hence $P = I - A^{(c)}{A^{(c)}}^g$ by Corollary 5.1.8. Let $y \in \mathrm{R}(P)$. Then $Py = y$, that is $(I - P)y = 0$, so that $A^{(c)}({A^{(c)}}^g y) = 0$. Now,

$$A^{(c)}y = (A^{(c)}{A^{(c)}}^g A^{(c)})y = A^{(c)}(A^{(c)}({A^{(c)}}^g y)) = A^{(c)}0 = 0.$$

Hence $y \in \mathrm{Null}(A^{(c)})$, so that $\mathrm{R}(P) \subseteq \mathrm{Null}(A^{(c)})$.

Conversely, suppose that $y \in \mathrm{Null}(A^{(c)})$. Then $A^{(c)}y = 0$, and hence ${A^{(c)}}^g(A^{(c)}y) = 0$, so that $(I - P)y = 0$. It then follows that $y = Py \in \mathrm{R}(P)$, and hence $\mathrm{Null}(A^{(c)}) \subseteq \mathrm{R}(P)$. The result then follows. ■

Corollary 7.1.5 *([13], (4.1)) Let $A \in \mathfrak{L}(X)^D$. Then $\mathrm{R}(A^{(c)}) = \mathrm{Null}(P)$, where P denotes the generalized Drazin idempotent of A.*

Proof:
Suppose that $A \in \mathfrak{L}(X)^D$. If $A \in \mathfrak{L}(X)^{-1}$, then $A = A^{(c)}$ and $P = 0$ by Theorem 5.1.6, so that the result clearly holds in this case. Suppose now that $0 \in \operatorname{iso}\sigma(A)$ and let P denote the spectral idempotent of A corresponding to 0. By Corollary 7.1.3, P is the spectral idempotent of $A^{(c)}$ corresponding to 0, and hence $P = I - A^{(c)}A^{(c)^g}$ by Corollary 5.1.8. Let $y \in \mathrm{R}(A^{(c)})$. Then there exists an $x \in X$ such that $A^{(c)}x = y$. Now,

$$Py = P(A^{(c)}x) = (I - A^{(c)}A^{(c)^g})(A^{(c)}x) = (A^{(c)} - A^{(c)}A^{(c)^g}A^{(c)})x = 0x = 0;$$

hence $y \in \mathrm{Null}(P)$, so that $\mathrm{R}(A^{(c)}) \subseteq \mathrm{Null}(P)$.

Conversely, suppose that $y \in \mathrm{Null}(P)$. Then $Py = 0$, and hence $(I - A^{(c)}A^{(c)^g})y = 0$, so that $y = A^{(c)}(A^{(c)^g}y)$. Let $x = A^{(c)^g}y$. Then $y = A^{(c)}x$, so that $y \in \mathrm{R}(A^{(c)})$. Hence $\mathrm{Null}(P) \subseteq \mathrm{R}(A^{(c)})$, so that the result follows. ■

The following result is an immediate consequence of Corollary 7.1.5 and will be needed in Section 7.3.

Corollary 7.1.6 *([13], p.174) Let $A \in \mathfrak{L}(X)^D$. The operator $A^{(c)}$ is a closed range operator.*

Proof:
Since the spectral idempotent P of A corresponding to 0 is a bounded linear operator, we have that $\mathrm{Null}(P)$ is closed. The result then follows from Corollary 7.1.5. ■

7.2 Further properties on the notion of the gap between closed subspaces

In order to discuss the continuity of the generalized Drazin inverse of a bounded linear operator, as was done in [13] by Koliha and Rakočević, additional properties of the gap are needed. In this section we give the properties of the gap presented in [13], where it was shown that these properties are satisfied by a special class of operators in $\mathfrak{L}(X)$, namely the idempotents. Take note that, for an idempotent $P \in \mathfrak{L}(X)$, since $\mathrm{Null}(P)$ is closed, we also have that $\mathrm{R}(P) = \mathrm{Null}(I - P)$ is closed. This fact is, however, not true for an arbitrary element in $\mathfrak{L}(X)$ ([15], Problem 6, p.101).

We state and prove the following result that will be required in the proof of Lemma 7.2.2.

Lemma 7.2.1 *Let $P, Q \in \mathfrak{L}(X)$ be idempotents. Then*

$$||I - P||\mathrm{D}(u, \mathrm{R}(P)) \leq ||I - P||\mathrm{gap}(\mathrm{R}(Q), \mathrm{R}(P))||u||,$$

for all $u \in \mathrm{R}(Q)$.

Proof:
Let $u \in \mathrm{R}(Q)$. If $u = 0$, then $u \in \mathrm{R}(P)$, and hence the result holds. If $u \in \mathrm{R}(Q)$ is such that $||u|| = 1$, then

$$\begin{aligned} ||I-P||\mathrm{D}(u, \mathrm{R}(P)) &\leq ||I-P||\delta(\mathrm{R}(Q), \mathrm{R}(P)) \\ &\leq ||I-P||\mathrm{gap}(\mathrm{R}(Q), \mathrm{R}(P)) \\ &= ||I-P||\mathrm{gap}(\mathrm{R}(Q), \mathrm{R}(P))||u||. \end{aligned}$$

If $0 \neq u \in \mathrm{R}(Q)$, then $\left|\left|\frac{u}{||u||}\right|\right| = 1$, and hence, by the previous reasoning,

$$||I-P||\mathrm{D}\left(\frac{u}{||u||}, \mathrm{R}(P)\right) \leq ||I-P||\mathrm{gap}(\mathrm{R}(Q), \mathrm{R}(P))\left|\left|\frac{u}{||u||}\right|\right|.$$

Since $\frac{1}{||u||} > 0$, we have that $\mathrm{D}\left(\frac{u}{||u||}, \mathrm{R}(P)\right) = \frac{1}{||u||}\mathrm{D}(u, \mathrm{R}(P))$, and hence the result follows. ∎

Recall that in Section 2.10 we mentioned that the gap can be seen as a function which measures the "distance" between two subspaces. Our next results gives upper and lower bounds for the "distances" between null spaces and ranges of idempotents in $\mathfrak{L}(X)$.

Lemma 7.2.2 *([13], Lemma 3.1) Let $P, Q \in \mathfrak{L}(X)$ be idempotents. Then*
(1) $\mathrm{gap}(\mathrm{R}(Q), \mathrm{R}(P)) \leq \max\{||(I-Q)P||, ||(I-P)Q||\}$,
(2) $\mathrm{gap}(\mathrm{Null}(Q), \mathrm{Null}(P)) \leq \max\{||Q(I-P)||, ||P(I-Q)||\}$,
(3) $||(I-P)Q|| \leq ||I-P||||Q||\mathrm{gap}(\mathrm{R}(Q), \mathrm{R}(P))$,
(4) $||P(I-Q)|| \leq ||P||||I-Q||\mathrm{gap}(\mathrm{Null}(Q), \mathrm{Null}(P))$.

Proof:
Suppose that P and Q are idempotents in $\mathfrak{L}(X)$.
(1) Let $u \in \mathrm{R}(P)$ with $||u|| = 1$. Then, since $Pu = u$, we have that

$$\begin{aligned} \mathrm{D}(u, \mathrm{R}(Q)) = \inf\{||u - Qx|| : x \in X\} &\leq ||u - Qu|| \\ &= ||(I-Q)u|| \\ &= ||(I-Q)Pu|| \\ &\leq \sup\{||(I-Q)Pu|| : ||u|| = 1\} \\ &= ||(I-Q)P||. \end{aligned}$$

Hence $\mathrm{D}(u, \mathrm{R}(Q)) \leq ||(I-Q)P||$, for all $u \in \mathrm{R}(P)$ with $||u|| = 1$, so that

$$\delta(\mathrm{R}(P), \mathrm{R}(Q)) = \sup\{\mathrm{D}(u, \mathrm{R}(Q)) : u \in \mathrm{R}(P), ||u|| = 1\} \leq ||(I-Q)P||.$$

By exchanging P and Q in the inequality above, we have that

$$\delta(\mathrm{R}(Q), \mathrm{R}(P)) = \sup\{\mathrm{D}(u, \mathrm{R}(P)) : u \in \mathrm{R}(Q), ||u|| = 1\} \leq ||(I-P)Q||.$$

Hence

$$\begin{aligned}\text{gap}(\text{R}(Q),\text{R}(P)) &= \max\{\delta(\text{R}(P),\text{R}(Q)),\delta(\text{R}(Q),\text{R}(P))\}\\ &\leq \max\{||(I-Q)P||,||(I-P)Q||\}.\end{aligned}$$

(2) Using (1) and the fact that $\text{Null}(P) = \text{R}(I-P)$ and $\text{Null}(Q) = \text{R}(I-Q)$, we have that

$$\begin{aligned}\text{gap}(\text{Null}(Q),\text{Null}(P)) &= \text{gap}(\text{R}(I-Q),\text{R}(I-P))\\ &\leq \max\{||Q(I-P)||,||P(I-Q)||\}.\end{aligned}$$

(3) If $I = P$, then statement (3) is true. Hence, suppose that $I \neq P$. Let $x \in X$ and $v \in \text{R}(P)$. Since $Pv = v$, we have that

$$(I-P)Qx = (I-P)Qx - (I-P)v = (I-P)(Qx-v).$$

Hence $||(I-P)Qx|| = ||(I-P)(Qx-v)|| \leq ||I-P||||Qx-v||$, for all $v \in \text{R}(P)$, so that $\frac{||(I-P)Qx||}{||I-P||} \leq ||Qx-v||$, for all $v \in \text{R}(P)$. It then follows that

$$\frac{||(I-P)Qx||}{||I-P||} \leq \inf\{||Qx-v|| : v \in \text{R}(P)\} = \text{D}(Qx,\text{R}(P)).$$

Using Lemma 7.2.1, we have that

$$\begin{aligned}||(I-P)Qx|| &\leq ||I-P||\text{D}(Qx,\text{R}(P))\\ &\leq ||I-P||\text{gap}(\text{R}(Q),\text{R}(P))||Qx||\\ &\leq ||I-P||\text{gap}(\text{R}(Q),\text{R}(P))||Q||||x||,\end{aligned}$$

and hence $||(I-P)Q|| \leq ||I-P||||Q||\text{gap}(\text{R}(Q),\text{R}(P))$.

(4) Using (3) we have that

$$\begin{aligned}||P(I-Q)||| &\leq ||P||||I-Q||\text{gap}(\text{R}(I-Q),\text{R}(I-P))\\ &= ||P||||I-Q||\text{gap}(\text{Null}(Q),\text{Null}(P)).\end{aligned}$$

■

The following result is an immediate consequence of Lemma 7.2.2.

Corollary 7.2.3 *([13], Corollary 3.2) Let $P, Q \in \mathfrak{L}(X)$ be commuting idempotents. Then*
(1) $||P-Q|| \leq (||I-P||||Q|| + ||P||||I-Q||)\text{gap}(\text{R}(Q),\text{R}(P))$ *and*
(2) $||P-Q|| \leq (||I-P||||Q|| + ||P||||I-Q||)\text{gap}(\text{Null}(Q),\text{Null}(P))$.

Proof:
Suppose that P and Q in $\mathfrak{L}(X)$ are commuting idempotents.

(1) Using Lemma 7.2.2(3) and the fact that $PQ = QP$, we have that

$$\begin{aligned} ||P - Q|| &= ||(P - QP) - (Q - PQ)|| \\ &\leq ||(I - Q)P|| + ||(I - P)Q|| \\ &\leq ||I - Q||||P||\text{gap}(\text{R}(P), \text{R}(Q)) + ||I - P||||Q||\text{gap}(\text{R}(Q), \text{R}(P)) \\ &= (||P||||I - Q|| + ||Q||||I - P||)\text{gap}(\text{R}(Q), \text{R}(P)). \end{aligned}$$

(2) Analogously, using Lemma 7.2.2(4) and the fact that $PQ = QP$, we have that

$$\begin{aligned} ||P - Q|| &= ||(P - QP) - (Q - PQ)|| \\ &\leq ||P(I - Q)|| + ||Q(I - P)|| \\ &\leq (||P||||I - Q|| + ||Q||||I - P||)\text{gap}(\text{Null}(Q), \text{Null}(P)). \end{aligned}$$

■

Our next result describes the convergence of idempotents in terms of the convergence of the gaps of null spaces and ranges.

Lemma 7.2.4 *([13], Lemma 3.3) If $P_n, P \in \mathfrak{L}(X)$ are idempotents, then the following statements are equivalent:*
(a) $||P_n - P|| \to 0$ *as* $n \to \infty$.
(b) $\text{gap}(\text{R}(P_n), \text{R}(P)) \to 0$ *and* $\text{gap}(\text{Null}(P_n), \text{Null}(P)) \to 0$.
Moreover, if $P_nP = PP_n$ for all sufficiently large n, then $\text{gap}(\text{R}(P_n), \text{R}(P)) \to 0$ *implies (a) and* $\text{gap}(\text{Null}(P_n), \text{Null}(P)) \to 0$ *implies (a).*

Proof:
Suppose that P and all P_n are idempotents in $\mathfrak{L}(X)$.
(a)⇒(b): If $||P_n - P|| \to 0$ as $n \to \infty$, then $||(I-P)P_n|| \to 0$, $||(I-P_n)P|| \to 0$, $||P_n(I - P)|| \to 0$ and $||P(I - P_n)|| \to 0$ as $n \to \infty$. Therefore,

$$\text{gap}(\text{R}(P_n), \text{R}(P)) \leq \max\{||(I - P_n)P||, ||(I - P)P_n||\} \to 0$$

and

$$\text{gap}(\text{Null}(P_n), \text{Null}(P)) \leq \max\{||P_n(I - P)||, ||P(I - P_n)||\} \to 0$$

as $n \to \infty$, when replacing Q by P_n in Lemma 7.2.2(1) and Lemma 7.2.2(2), respectively.

(b)⇒(a): Suppose that $\text{gap}(\text{R}(P_n), \text{R}(P)) \to 0$ and $\text{gap}(\text{Null}(P_n), \text{Null}(P)) \to 0$ as $n \to \infty$. Let

$$\mu(P_n) = \max\{||P_n||, ||I - P_n||\} \leq 1 + ||P_n||. \tag{7.2.1}$$

By replacing Q by P_n in Lemma 7.2.2[(3) and (4)], we have that

$$\begin{aligned}
||P_n - P|| &= ||P_n - PP_n - (P - PP_n)|| \\
&\leq ||(I-P)P_n|| + ||P(I-P_n)|| \\
&\leq ||I-P||||P_n||\text{gap}(\text{R}(P_n), \text{R}(P)) \\
&\qquad\qquad + ||P||||I-P_n||\text{gap}(\text{Null}(P_n), \text{Null}(P)) \\
&\leq \mu(P_n)[||I-P||\text{gap}(\text{R}(P_n), \text{R}(P)) + ||P||\text{gap}(\text{Null}(P_n), \text{Null}(P))].
\end{aligned}$$

Let

$$\delta_n = ||I-P||\text{gap}(\text{R}(P_n), \text{R}(P)) + ||P||\text{gap}(\text{Null}(P_n), \text{Null}(P)).$$

By assumption $\delta_n \to 0$ as $n \to \infty$. Hence, we can find an $N \in \mathbb{N}$ such that $0 \leq \delta_n < \frac{1}{2}$, for all $n \geq N$. Also, $||P_n - P|| \leq \mu(P_n)\delta_n \leq (1 + ||P_n||)\delta_n$ by equation (7.2.1). Now,

$$\begin{aligned}
||P_n|| = ||P + P_n - P|| &\leq ||P|| + ||P_n - P|| \\
&\leq ||P|| + (1 + ||P_n||)\delta_n \\
&= ||P|| + \delta_n + ||P_n||\delta_n,
\end{aligned}$$

so that $(1-\delta_n)||P_n|| \leq ||P|| + \delta_n$, that is $||P_n|| \leq \frac{||P||+\delta_n}{1-\delta_n} < 2||P|| + 1$; hence

$$1 + ||P_n|| < 2 + 2||P|| \tag{7.2.2}$$

for all $n \geq N$. It then follows that

$$||P_n - P|| \leq (1 + ||P_n||)\delta_n < (2||P|| + 2)\delta_n \to 0 \text{ as } n \to \infty.$$

This completes the first part of the lemma.

Suppose now that P_n and P are commuting idempotents for large n. Combining (7.2.1) and (7.2.2), we have that $\mu(P_n) = \max\{||P_n||, ||I - P_n||\}$ is bounded by $2 + 2||P||$ for all $n \geq N$. By also using Corollary 7.2.3, with $Q = P_n$, we have, for all $n \geq \mathbb{N}$, that

$$\begin{aligned}
||P - P_n|| &\leq (||I-P||||P_n|| + ||P||||I-P_n||)\text{gap}(\text{R}(P_n), \text{R}(P)) \\
&\leq (2 + 2||P||)[||I-P|| + ||P||]\text{gap}(\text{R}(P_n), \text{R}(P))
\end{aligned}$$

and

$$\begin{aligned}
||P - P_n|| &\leq (||I-P||||P_n|| + ||P||||I-P_n||)\text{gap}(\text{Null}(P_n), \text{Null}(P)). \\
&\leq (2 + 2||P||)[||I-P|| + ||P||]\text{gap}(\text{Null}(P_n), \text{Null}(P)).
\end{aligned}$$

It is then clear that if either of the two conditions in (b) holds, then (a) holds.

■

7.3 Continuity of the generalized Drazin inverse of a bounded linear operator

In [24], Rakočević studied the continuity of the Drazin inverse of a bounded linear operator. In this section we present some continuity properties, as was done in [13] by Koliha and Rakočević, of generalized Drazin invertible operators. The results obtained for the gap and reduced minimum modulus (see Section 2.11) will play a vital role in providing more characterizations for continuity of generalized Drazin inversion in $\mathfrak{L}(X)$. The main result in this section is Theorem 7.3.3.

In order to prove Theorem 7.3.3, the following result is needed. Rakočević showed this result for Drazin invertible bounded linear operators in ([24], Lemma 2.1). In Lemma 7.3.1 we formulate and prove this result for generalized Drazin invertible bounded linear operators. Let us mention that the proof presented here is analogous to the proof given by Rakočević.

Lemma 7.3.1 *([24], Lemma 2.1) Let $A \in \mathfrak{L}(X)^D$. Then*

$$\mathcal{B}'(0, \gamma(A^{(c)})) \subseteq \rho(A).$$

Proof:
Suppose that $A \in \mathfrak{L}(X)^D$. By Corollary 7.1.6, $\mathrm{R}(A^{(c)})$ is closed and hence a Banach space. We also have that $\gamma(A^{(c)}) > 0$ by Theorem 2.9.5.
If $A \in \mathfrak{L}(X)^{-1}$ and $\lambda \in \mathcal{B}(0, \gamma(A^{(c)}))$, then, by using Lemma 2.9.4 and the fact that $A^{(c)} = A$, it follows that

$$||(A - \lambda I) - A|| = |\lambda| < \gamma(A^{(c)}) = \gamma(A) = ||A^{-1}||^{-1},$$

and hence $\lambda I - A \in \mathfrak{L}(X)^{-1}$ by Theorem 2.7.1, so that $\lambda \in \rho(A)$. This completes the proof for the case where A is invertible.

Now, let $A \in \mathfrak{L}(X)^D \backslash \mathfrak{L}(X)^{-1}$. Since $A^{(c)} \in \mathfrak{L}(X)^g$ by Theorem 7.1.1, we have from Theorem 2.12.6 that $\mathrm{asc}(A^{(c)}) = \mathrm{des}(A^{(c)}) = 1$, and hence $X = \mathrm{Null}(A^{(c)}) \oplus \mathrm{R}(A^{(c)})$ by Theorem 2.8.2. We also have from Theorem 2.8.2 that $A_0^{(c)} = A^{(c)}_{|\mathrm{Null}(A^{(c)})} = 0$ and that $A_1^{(c)} = A^{(c)}_{|\mathrm{R}(A^{(c)})}$ is invertible in $\mathfrak{L}(\mathrm{R}(A^{(c)}))$. Let $I_0 = I_{|\mathrm{Null}(A^{(c)})}$ and $I_1 = I_{|\mathrm{R}(A^{(c)})}$. Since $A_0^{(c)}$ is nilpotent, it is quasinilpotent, so that

$$\lambda I_0 - A_0^{(c)} \in \mathfrak{L}(\mathrm{Null}(A^{(c)}))^{-1} \tag{7.3.1}$$

for all $\lambda \neq 0$. Now, if $\lambda \in \mathcal{B}(0, \gamma(A^{(c)}))$, then by using Lemma 2.9.4, we have that

$$||(A_1^{(c)} - \lambda I_1) - A_1^{(c)}|| = |\lambda| < \gamma(A^{(c)}) \leq \gamma(A_1^{(c)}) = \left|\left|A_1^{(c)^{-1}}\right|\right|^{-1},$$

so that

$$\lambda I_1 - A_1^{(c)} \in \mathfrak{L}(\mathrm{R}(A^{(c)}))^{-1} \tag{7.3.2}$$

by Theorem 2.7.1. Combining (7.3.1) and (7.3.2), we have that $\lambda I - A^{(c)} \in \mathfrak{L}(X)^{-1}$ for $\lambda \in \mathcal{B}'(0, \gamma(A^{(c)}))$, which implies that $\mathcal{B}'(0, \gamma(A^{(c)})) \subseteq \rho(A^{(c)})$. By recalling Corollary 7.1.2 the proof is completed. ∎

Corollary 7.3.2 *Let $A \in \mathfrak{L}(X)^D$. Then $\gamma(A^{(c)}) \leq \mathrm{D}(0, \sigma'(A))$.*

Proof:
Let $\lambda \in \sigma'(A)$. By Lemma 7.3.1 we have that $\lambda \notin \mathcal{B}'(0, \gamma(A^{(c)}))$, so that $|\lambda| \geq \gamma(A^{(c)})$, and hence

$$\mathrm{D}(0, \sigma'(A)) = \inf\{|\lambda| : \lambda \in \sigma'(A)\} \geq \gamma(A^{(c)}).$$

∎

We are now ready to present the main result in this section. This result is due to Koliha and Rakočević (see [13]).

Theorem 7.3.3 *([13], Theorem 4.1) Let $A_n, A \in \mathfrak{L}(X)^D$ be such that (A_n) is a convergent sequence with limit $A \in \mathfrak{L}(X)$. Let P_n and P be the generalized Drazin idempotents of A_n and A, respectively. Then the following statements are equivalent:*
(a) $A_n^D \to A^D$ *as* $n \to \infty$.
(b) $\sup\{||A_n^D|| : n \in \mathbb{N}\} < \infty$.
(c) $\sup\{r(A_n^D) : n \in \mathbb{N}\} < \infty$.
(d) $\inf\{\mathrm{D}(0, \sigma'(A_n)) : n \in \mathbb{N}\} > 0$.
(e) There exists an $r > 0$ *such that* $\mathcal{B}'(0, r) \subseteq \rho(A) \cap \left(\bigcap_{n=1}^{\infty} \rho(A_n)\right)$.
(f) $A_n^D A_n \to A^D A$ *as* $n \to \infty$.
(g) $P_n \to P$ *as* $n \to \infty$.
(h) $\mathrm{gap}(\mathrm{R}(P_n), \mathrm{R}(P)) \to 0$ *and* $\mathrm{gap}(\mathrm{Null}(P_n), \mathrm{Null}(P)) \to 0$ *as* $n \to \infty$.
(i) $\mathrm{gap}(\mathrm{R}(A_n^{(c)}), \mathrm{R}(A^{(c)})) \to 0$ *and* $\mathrm{gap}(\mathrm{Null}(A_n^{(c)}), \mathrm{Null}(A^{(c)})) \to 0$ *as* $n \to \infty$.
(j) $A_n^{(c)} \to A^{(c)}$ *and* $\mathrm{gap}(\mathrm{R}(A_n^{(c)}), \mathrm{R}(A^{(c)})) \to 0$ *as* $n \to \infty$.
(k) $A_n^{(c)} \to A^{(c)}$ *and* $\mathrm{gap}(\mathrm{Null}(A_n^{(c)}), \mathrm{Null}(A^{(c)})) \to 0$ *as* $n \to \infty$.
(l) $\gamma(A_n^{(c)}) \to \gamma(A^{(c)})$ *as* $n \to \infty$.
(m) $\inf\{\gamma(A_n^{(c)}) : n \in \mathbb{N}\} > 0$.

Proof:
Suppose that $A_n \to A$ as $n \to \infty$, where A and all A_n are generalized Drazin invertible in $\mathfrak{L}(X)$. Denote by P_n and P the generalized Drazin idempotents of A_n and A, respectively.

The equivalences (a) through (g) follow when we apply the results obtained in Theorem 6.2.13 in the Banach algebra setting to the bounded linear operator case.

The statements (g) and (h) are equivalent by Lemma 7.2.4.
Using Corollaries 7.1.4 and 7.1.5, we have that (h)⇔(i).
Suppose (i) holds. Using Corollaries 5.3.2 and 5.3.3 we have that $A_n^{(c)} = A_n(I - P_n)$ and $A^{(c)} = A(I - P)$. Since (g)⇔(i), we have that $P_n \to P$ as $n \to \infty$, and hence, by the continuity of multiplication in a Banach algebra, it follows that $A_n^{(c)} \to A^{(c)}$ as $n \to \infty$. Hence (i)⇒(j) and (i)⇒(k).
By Corollary 7.1.6, the core operators are closed range operators. The statements (j) and (k) are then equivalent by Lemma 2.11.4.
We also have by Lemma 2.11.4 that the implication (k)⇒(l) holds.
By Corollary 7.1.6 and the proof of Lemma 2.11.4 ((b)⇒(a)) it is clear that (l)⇒(m).
Suppose that (m) holds. By Corollary 7.3.2, $\gamma(A_n^{(c)}) \leq \mathrm{D}(0, \sigma'(A_n))$ for all $n \in \mathbb{N}$. Hence $0 < \inf\{\gamma(A_n^{(c)}) : n \in \mathbb{N}\} \leq \inf\{\mathrm{D}(0, \sigma'(A_n)) : n \in \mathbb{N}\}$, so that (m)⇒(d). The proof of the theorem is complete. ■

Remark 7.3.4 *If Theorem 7.3.3 is specialized to operators in $\mathfrak{L}(X)^d$, then we get the result of Rakočević in ([24], Theorem 2.2). In the generalized Drazin invertible bounded linear operator case, Koliha and Rakočević succeeded in imposing a number of new equivalent conditions.*

In ([24], Corollary 3.3) Rakočević showed that in the case of Drazin invertible bounded linear operators (assuming that the Drazin degrees are uniformly bounded) the condition $A_n^{(c)} \to A^{(c)}$ in Theorem 7.3.3[(j) and (k)] can be omitted. A natural question would be whether this is also possible for generalized Drazin invertible bounded linear operators. In ([13], Note 4.3) the authors mentioned a few cases under which it is possible to reduce (j) and (k) in Theorem 7.3.3 to $\mathrm{gap}(\mathrm{R}(A_n^{(c)}), \mathrm{R}(A^{(c)})) \to 0$ and $\mathrm{gap}(\mathrm{Null}(A_n^{(c)}), \mathrm{Null}(A^{(c)})) \to 0$ as $n \to \infty$ alone, respectively. However, from Lemma 2.11.3 [(1) and (2)] it appears that this might not be possible in general.

It is also a logical question to ask whether the conditions $\mathrm{gap}(\mathrm{R}(A_n^{(c)}), \mathrm{R}(A^{(c)})) \to 0$ and $\mathrm{gap}(\mathrm{Null}(A_n^{(c)}), \mathrm{Null}(A^{(c)})) \to 0$ as $n \to \infty$ are essential in (j) and (k) of Theorem 7.3.3, respectively. It is indeed the case, as illustrated in the next example.

Example 7.3.5 *([13], Example 4.2) In general, it is not possible to omit $\mathrm{gap}(\mathrm{R}(A_n^{(c)}), \mathrm{R}(A^{(c)})) \to 0$ and $\mathrm{gap}(\mathrm{Null}(A_n^{(c)}), \mathrm{Null}(A^{(c)})) \to 0$ as $n \to \infty$ from (j) and (k) in Theorem 7.3.3.*

Define $A_n : \mathbb{R}^3 \to \mathbb{R}^3$ and $A : \mathbb{R}^3 \to \mathbb{R}^3$ by $A_n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{n} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, respectively. Then $A_n, A \in \mathfrak{L}(\mathbb{R}^3)$. Moreover, since A is idempotent, it is group invertible with $A^g = A = A^{(c)}$ by Theorem 7.1.1; hence

$$A^D = A^g = A^{(c)g} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $\operatorname{rank}(A_n) = \operatorname{rank}(A_n^2)$ for all $n \in \mathbb{N}$, we have from Theorem 2.12.2 that all A_n are group invertible, and hence $A_n^{(c)} = A_n$ by Theorem 7.1.1, so that

$$A_n^D = A_n^g = A_n^{(c)g} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is clear that $A_n \to A$ as $n \to \infty$, and hence $A_n^{(c)} \to A^{(c)}$ as $n \to \infty$, but (A_n^D) does not converge to A^D, in fact, (A_n^D) diverges. ■

List of References

[1] B. Aupetit. *Propriétés spectrales des algèbres de Banach*, volume 735 of *Lecture Notes in Math.* Springer-Verlag, New York, 1979.

[2] B. Aupetit. *A primer on spectral theory.* Springer-Verlag, New York, 1991.

[3] B. Aupetit and H. du T. Mouton. Trace and determinant in Banach algebras. *Studia Math.*, **121**(**2**):115–136, 1996.

[4] A. Ben-Israel and T. N. E. Greville. *Generalized inverses: Theory and applications.* Springer, Berlin, second edition, 2003.

[5] F. F. Bonsall and J. Duncan. *Complete normed algebras*, volume (Band) 80 of *Ergeb. Math. Grenzgeb.* Springer-Verlag, Berlin, 1973.

[6] R. M. Brits, L. Lindeboom, and H. Raubenheimer. On ideals of generalized invertible elements in Banach algebras. *Math. Proc. R. Ir. Acad.*, **105A**(**2**):1–10, 2005.

[7] R. M. Brits, L. Lindeboom, and H. Raubenheimer. Rank and the Drazin inverse in Banach algebras. *Studia Math.*, **177**(**3**):211–224, 2006.

[8] S. L. Campbell and C. D. Meyer. Continuity properties of the Drazin pseudoinverse. *Linear Algebra Appl.*, **10**(**1**):77–83, 1975.

[9] M. P. Drazin. Pseudo-inverses in associative rings and semigroups. *Amer. Math. Monthly*, **65**(**7**):506–514, 1958.

[10] R. E. Harte. *Invertibility and singularity for bounded linear operators*, volume 109 of *Monographs and Textbooks in Pure and Applied Mathematics.* Marcel Dekker, New York, 1988.

[11] C. F. King. A note on Drazin inverses. *Pacific J. Math.*, **70**(**2**):383–390, 1977.

[12] J. J. Koliha. A generalized Drazin inverse. *Glasgow Math. J.*, **38**:367–381, 1996.

[13] J. J. Koliha and V. Rakočević. Continuity of the Drazin inverse II. *Studia Math.*, **131**(**2**):167–177, 1998.

[14] B. Kolman. *Introductory linear algebra with applications.* John Wiley and Sons, New York, fifth edition, 1989.

[15] E. Kreyszig. *Introductory functional analysis with applications.* John Wiley and Sons, New York, 1978.

[16] A. S. Markus. *On some properties of linear operators connected with the notion of the gap*, volume 39 of *Unchen. Zap. Kishinev. Gos. Univ.* 1959.

[17] R. E. Megginson. *An introduction to Banach space theory*, volume 183 of *Graduate texts in Math.* Springer-Verlag, New York, 1998.

[18] T. Mouton and H. Raubenheimer. On rank one and finite elements of Banach algebras. *Studia Math.*, **104**(**3**):211–219, 1993.

[19] V. Müller. *Spectral theory of linear operators and spectral systems in Banach algebras*, volume 139 of *Operator Theory: Advances and Applications.* Birkhäuser, Basel, 2007.

[20] K. Muzundu. Rank one and finite rank elements of Banach algebras. Masters thesis, Stellenbosch University, 2007.

[21] T. W. Palmer. *Banach algebras and the general theory of $*$-algebras*, volume 1 Algebras and Banach algebras of *Encyclopedia Math. Appl. 49.* Cambridge Univ. Press, Cambridge, 1994.

[22] R. Penrose. A generalized inverse for matrices. *Proc. Cambridge Philos. Soc.*, **51**:406–413, 1955.

[23] J. Puhl. The trace of finite and nuclear elements in Banach algebras. *Czechoslovak Math. J.*, **28**(**103**):656–676, 1978.

[24] V. Rakočević. Continuity of the Drazin inverse. *J. Operator Theory*, **41**:55–68, 1999.

[25] S. Roch and B. Silbermann. Continuity of generalized inverses in Banach algebras. *Studia Math.*, **136**(**3**):197–227, 1999.

Index